1. Introduction

Let $X$ be a separable infinite dimensional real Banach space. There are three general types of questions we often ask.

# 1. What can be said about the structure of $X$ itself?
# 2. Can one find a “nice” subspace $Y \subseteq X$?
# 3. Does $X$ embed into a “nice” superspace $X \subseteq Z$?

$X$ could be arbitrary, $X$ might have some specific property or it might be completely defined, e.g., $X = \ell_p$ ($1 \leq p < \infty$), $X = c_0$, $X = L_p$ ($1 \leq p < \infty$), $X = C[0, 1]$ or $X = C(K)$, $K = \text{compact metric}$,
the standard Banach spaces one first encounters when studying functional analysis.

**Notation.** $X, Y, Z, \ldots$ will always denote separable infinite dimensional real Banach spaces unless specified otherwise. $X$ is *isomorphic* to $Y$ ($X \sim Y$) if there exists a one-to-one bounded linear operator $T : X \onto Y$ (hence $T^{-1} : Y \to X$ is bounded). Thus there exist $0 < c, C < \infty$ such that for all $x \in X$

$$c\|x\| \leq \|Tx\| \leq C\|x\|$$

in which case we write $X^K \sim Y$ where $K = Cc^{-1}$. We write $X \hookrightarrow Y$ if $X^K \sim Z$ for some $Z \subseteq Y$.

The *Banach-Mazur distance* between isomorphic spaces $X$ and $Y$ is $d(X, Y) = \inf\{K : X^K \sim Y\}$.

**Caution.** To get a metric you need to take $\log(d(X, Y))$ and identify $X$ with all $Y$’s satisfying $d(X, Y) = 1$. The Banach-Mazur distance satisfies $d(X, Z) \leq d(X, Y)d(Y, Z)$ rather than the triangle inequality if $X \sim Y \sim Z$.

We first consider our three questions in terms of a basis, as an illustration.
**Definition.** $(e_i)_{i=1}^\infty$ is a basis for $X$ if for all $x \in X$ there exists a unique sequence of reals $(a_i)_{i=1}^\infty$ so that

$$x = \sum_{i=1}^\infty a_i e_i .$$

**Proposition 1.1.** Let $(e_i)_{i=1}^\infty \subseteq X$. Then $(e_i)$ is a basis for $X$ iff

i) $e_i \neq 0$ for all $i$

ii) $[(e_i)] \equiv$ the closed linear span of $(e_i) = X$

iii) There exists $K < \infty$ so that for all $n < m$ in $\mathbb{N}$ and $(a_i)_{1}^{m} \subseteq \mathbb{R}$

$$\left\| \sum_{1}^{n} a_i e_i \right\| \leq K \left\| \sum_{1}^{m} a_i e_i \right\|$$

In this case the smallest such $K$ is called the **basis constant** of $(e_i)$ and we say $(e_i)$ is $K$-basic.

$(e_i)$ is **monotone** if $K = 1$.

$(e_i) \subseteq X$ is **basic** if $(e_i)$ is a basis for $[(e_i)]$.

Thus if $(e_i)$ is a basis for $X$ the basis projections $(P_n)_{n=1}^\infty$, given by $P_n(\sum_{1}^{\infty} a_i e_i) = \sum_{1}^{n} a_i e_i$ are uniformly bounded and the basis constant of $(e_i)$ is $\sup_n \|P_n\|$. From this we obtain that the coordinate functionals $(e_i^*)$, given by $e_i^*(\sum a_j e_j) = a_i$, are elements of $X^*$. For every $x \in X$, $x = \sum_{1}^{\infty} e_i^*(x)e_i$. 
Sometimes it is more convenient to use the projection constant of a basis \((e_i)\) given by \(\sup_{n \leq m} \|P_{[n,m]}\|\), where

\[
P_{[n,m]}\left(\sum a_i e_i\right) = \sum_{i=n}^{m} a_i e_i.
\]

\((e_i)\) is bimonotone if this constant is 1.

It is worth noting that if \((e_i)\) is a basis for \(X\) then we can renorm \(X\) so that \((e_i)\) is monotone, by setting

\[
\left\| \sum a_i e_i \right\| = \sup_n \left\| P_n\left(\sum a_i e_i\right) \right\|
\]

or even bimonotone by setting

\[
\left\| \sum a_i e_i \right\| = \sup_{n \leq m} \left\| P_{[n,m]}\left(\sum a_i e_i\right) \right\|.
\]

By “renorming” \(X\) we mean that \(\| \cdot \|\) is an equivalent norm on \(X\), i.e., for some \(c, C\) in \((0, \infty)\) and all \(x \in X\),

\[
c\|x\| \leq \|x\| \leq C\|x\|.
\]

Note that if \((e_i)\) is a normalized basis for \(X\) and \(K\) is the projection constant of \((e_i)\) then for all scalars \((a_i)\),

\[
K^{-1}\sup_i |a_i| \leq \left\| \sum a_i e_i \right\| \leq \sum_i |a_i|.
\]

In other words the space \(X\) sits between the \(c_0\) and \(\ell_1\) norms.

All of the classical spaces listed above have monotone bases. The unit vector basis \((e_i)\) given by \(e_i(j) = \delta_{i,j}\)
is a basis for $\ell_p$, $1 \leq p < \infty$ and for $c_0$. The **Haar basis** $(h_i)$ is a monotone basis for $L_p$, $1 \leq p < \infty$, and the **Schauder basis** $(f_i)$ is a monotone basis for $C[0, 1]$.

Basic sequences $(x_i)$ and $(y_i)$ are **$K$-equivalent** if for some $c^{-1}C \leq K$

$$c \left\| \sum a_iy_i \right\| \leq \left\| \sum a_ix_i \right\| \leq C \left\| \sum a_iy_i \right\|$$

for all scalars $(a_i)$. We denote this by $(x_i) \overset{K}{\sim} (y_i)$. Equivalently $T : [(x_i)] \rightarrow [(y_i)]$, given by $Tx_i = y_i$ for all $i$, is an isomorphism with $\|T\| \|T^{-1}\| \leq K$. We let

$$d_b((x_i), (y_i)) = \|T\| \|T^{-1}\|$$

define the “basis distance” between $(x_i)$ and $(y_i)$.

If $(e_i)$ is a basis for $X$, a **block basis** of $(e_i)$ is a nonzero sequence $(x_i)_{i=1}^\infty$ given by

$$x_i = \sum_{j=n_{i-1}+1}^{n_i} a_j e_j$$

for some sequence of integers $0 = n_0 < n_1 < \cdots$ and scalars $(a_i)$. $(x_i)$ is necessarily basic with basis constant not exceeding the basis constant of $(e_i)$.

**Perturbations** of basic sequences are basic sequences equivalent to the original.
Proposition 1.2. Let \((y_i)\) be a normalized basic sequence in \(X\) and let \((x_i)\) satisfy
\[
\sum_{i=1}^{\infty} \|y_i - x_i\| = \lambda < \frac{1}{2K}
\]
where \(K\) is the basis constant of \((y_i)\). Then \((x_i)\) is basic and \((x_i) \overset{C(\lambda)}{\sim} (y_i)\) where \(C(\lambda)\) approaches 1 as \(\lambda \downarrow 0\).

We will have more preliminary remarks shortly but first let us examine our three general questions in terms of “nice” meaning “has a basis”.

P. Enflo \([E1]\) proved in 1973 that not every \(X\) has a basis (see \([LT1]\), p.29). But, as known to Banach \([B]\), every \(X \overset{1}{\hookrightarrow} C[0,1]\) which has a basis. Indeed \(K = (B_{X^*}, \omega^*)\), the unit dual ball in the \(\omega^*\)-topology, is compact metric. Every compact metric space is a continuous image of the Cantor set \(\Delta\), say \(f : \Delta \rightarrow K\) is a continuous surjection. Define \(T : X \rightarrow C(\Delta)\) by \(Tx = x|_K \circ f\) to get an into isometry of \(X\) into \(C(\Delta)\). But \(C(\Delta) \overset{1}{\hookrightarrow} C[0,1]\) by an easy extension argument. So question \# 1 has a negative answer for “basis” and \# 3 has a positive one. Every \(X\) contains a basic sequence so \# 2 also has a positive answer. Indeed we may regard \(X \subseteq C[0,1]\) and then it is easy
to construct a sequence \((x_n) \subseteq S_X\), the unit sphere of \(X\), with for all \(m, x_m \in [(f_i)_{i=m}^\infty]\), where \((f_i)\) is the Schauder basis for \(C[0,1]\). Hence a subsequence \((y_i)\) of \((x_i)\) is a perturbation of a block basis of \((f_i)\) and hence is basic. In fact given \(\varepsilon > 0\) one can get \((y_i)\) to be \(1 + \varepsilon\)-basic.

It is worth noting that by similar arguments we have that

- If \(X \subseteq Y\) and \(Y\) has a basis \((y_i)\) then \(X\) contains a basic sequence \((x_i)\) which is a perturbation of a normalized block basis of \((y_i)\).
- If \((x_i)\) is normalized weakly null in \(X\) then a subsequence is \(1 + \varepsilon\)-basic and if \(X \subseteq Y\) as above it is also a perturbation of a normalized block basis of \((y_i)\).

While not every \(X\) has a basis, \(X\) always possesses a weaker structure.

**Theorem 1.3** ([OP], [Pe2]). *For all \(X\) and \(\varepsilon > 0\) there exists a biorthogonal system \((x_n, x_n^*)^\infty \subseteq X \times X^*\) with \(\|x_n\| = 1\) and \(\|x_n^*\| < 1 + \varepsilon\) so that \([x_n] = X\) and \(\langle x_n^*\rangle \equiv \text{the linear span of } (x_n^*)\) is \(\omega^*\)-dense in \(X^*\).*

So we can always find some sort of weak coordinate system.
In general not much can be said in regard to \# 1 for an arbitrary $X$ and not much more can be said about \# 3. We shall have to specify more structure to get results in these cases. \# 2 was the source of much research which we will discuss later.

A basis $\left( e_i \right)$ is $K$-unconditional if for all scalars $(a_i)$ and all $\varepsilon_i = \pm 1$

$$\left \| \sum \varepsilon_i a_i e_i \right \| \leq K \left \| \sum a_i e_i \right \|.$$ 

The smallest such $K$ is the unconditional basis constant of $\left( e_i \right)$. This is equivalent to the existence of $C$ so that for all scalars $(a_i)$ and $F \subseteq \mathbb{N}$

$$\left \| \sum_{F} a_i e_i \right \| \leq C \left \| \sum a_i e_i \right \|.$$ 

The smallest such $C$ is the suppression unconditional constant of $\left( e_i \right)$ and $\left( e_i \right)$ is called $C$-suppression unconditional. Easily, if $\left( e_i \right)$ is $K$-unconditional it is $K$-suppression unconditional and if it is $C$-suppression unconditional it is $2C$-unconditional. One can show

**Proposition 1.4.** Let $\left( e_i \right)$ be basic. Then $\left( e_i \right)$ is unconditional iff whenever $\sum a_i e_i = x$ and $\pi$ is any permutation of $\mathbb{N}$ then $\sum a_{\pi(i)} e_{\pi(i)} = x$.

In other words the convergence of $\sum a_i e_i$ to $X$ is “unconditional”.
Clearly the unit vector basis \((e_i)\) is a 1-unconditional basis for \(\ell_p\) \((1 \leq p < \infty)\) and for \(c_0\). The Haar basis \((h_i)\) is an unconditional basis for \(L_p\) iff \(1 < p < \infty\) (a more difficult result, see [Bu1]). The results above on block bases and perturbations pass to unconditional bases.

**Proposition 1.5.** Let \(X\) have an unconditional basis. Then \(X\) contains a subspace \(Y\) isomorphic to \(c_0\) or \(\ell_1\) or else \(X\) is reflexive.

The proof involves two important notions.

**Definition.** Let \((e_i)\) be a basis for \(X\).

\(\textbf{a)}\) \((e_i)\) is **boundedly complete** if whenever \((a_i) \subseteq \mathbb{R}\) satisfies \(\sup_n \| \sum_1^n a_i e_i \| < \infty\) then \(\sum_1^\infty a_i e_i\) converges.

\(\textbf{b)}\) \((e_i)\) is **shrinking** if every normalized block basis \((x_i)\) of \((e_i)\) is weakly null (i.e., \(x^*(x_i) \to 0\) for all \(x^* \in X^*\)).

The unit vector basis \((e_i)\) for \(\ell_p\) \((1 < p < \infty)\) is both shrinking and boundedly complete. In \(\ell_1\) it is boundedly complete but not shrinking and in \(c_0\) it is shrinking but not boundedly complete.
If \((e_i)\) is a basis for \(X\) then the biorthogonal (or co-ordinate functionals) \((e_i^*)\) are a basic sequence in \(X^*\) with basis constant not exceeding that of \((e_i)\). They also form a \(\omega^*\)-basis for \(X^*\); for all \(x^* \in X^*\) there exist unique scalars \((a_i)\) with \(x^* = \omega^* - \lim_{n \to \infty} \sum_{i=1}^{n} a_i e_i^*\). Of course, \(a_i = x^*(e_i)\). One can show that a basis \((e_i)\) for \(X\) is shrinking iff \([(e_i^*)] = X^*\), and so \((e_i^*)\) is actually a basis for \(X^*\).

It is also not hard to show that if \((e_i)\) is a boundedly complete basis for \(X\) then \(X\) is isomorphic to a dual space, namely \(Y^*\) where \(Y = [(e_i^*)] \subseteq X^*\).

If \((e_i)\) is a shrinking basis for \(X\) then \((e_i^*)\) is a boundedly complete basis for \(X^*\). If \((e_i)\) is a boundedly complete basis for \(X\) then \((e_i^*)\) is a shrinking basis for \([(e_i^*)]\).

\(X\) is reflexive iff \(B_X\) is weakly compact iff \(B_X\) is weakly sequentially compact (i.e., for all bounded \((x_i) \subseteq X\) there is a subsequence \((y_i)\) and \(y \in X\) with \(y_i \omega \to y\), i.e., \(x^*(y_i) \to x^*(y)\) for all \(x^* \in X^*\)). To prove Proposition 1.4 we have James’ result [J1].

**Proposition 1.6.** Let \(X\) have a basis \((e_i)\). \(X\) is reflexive iff \((e_i)\) is boundedly complete and shrinking.

Indeed if \((e_i)\) is boundedly complete and shrinking and \((x_i) \subseteq B_X\) then passing to a subsequence we may assume \(\lim_i e_j^*(x_i) \equiv a_j\) exists for all \(j\) and moreover
\[ \sup_n \| \sum_{j=1}^n a_j e_j \| < \infty \] so \( x \equiv \sum_1^\infty a_j e_j \in X \). It follows from \( (e_i) \) is shrinking that \( x_i - x \stackrel{\omega}{\to} 0 \) so \( x_i \stackrel{\omega}{\to} x \). The converse is also easy.

**Proof of Proposition 1.4.** Let \( (e_i) \) be an unconditional basis for \( X \). If \( (e_i) \) is not boundedly complete we obtain a seminormalized block basis \( (x_i) \), i.e., \( 0 < \inf \| x_i \| \leq \sup \| x_i \| < \infty \), of \( (e_i) \) with

\[
\sup_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\| \equiv K < \infty
\]

It follows easily that for all \( x^* \in B_{[(x_i)]^*}, \sum_1^\infty |x^*(x_i)| \leq K \). Thus if \( \|x^*\| = 1 \) and \( x^* \) norms \( \sum a_i x_i \) we have, \( \| \sum_1^\infty a_i x_i \| = x^* (\sum_1^\infty a_i x_i) \leq K \sup |a_i| \), so we see that \( (x_i) \) is equivalent to the unit vector basis of \( c_0 \).

If \( (e_i) \) is not shrinking we obtain a normalized block basis \( (x_i) \) and \( x^* \in B_{X^*} \) with \( x^*(x_i) > \lambda > 0 \) for all \( i \). Since \( (x_i) \) is unconditional, say with unconditional constant \( C \),

\[
\lambda \sum |a_i| \leq x^* \left( \sum \text{sign } a_i a_i x_i \right) \leq C \left\| \sum a_i x_i \right\| \leq C \sum |a_i|
\]

and so \( (x_i) \) is equivalent to the unit vector basis of \( \ell_1 \).

In terms of “nice” meaning our space has an unconditional basis, our questions \# 2 and \# 3 become
# 2. Does every $X$ contain an unconditional basic sequence?
   (No – [Gowers, Maurey 1993]; more about this later.)

# 3. Does every $X$ embed into a $Y$ with an unconditional basis?

This is easily shown to be false in a number of ways. We will show it in an unorthodox manner for future purposes.

If $(x_i)$ is a normalized weakly null sequence in a space with an unconditional basis then a subsequence of $(x_i)$ is unconditional by our earlier remarks. We shall show this is false, in general, and thus deduce that $C[0, 1]$ does not embed into a space with an unconditional basis.

The proof will also be an illustration of how to construct new Banach spaces.

**Example 1.7.** [MR] There exists a normalized weakly null sequence with no unconditional subsequence.

We shall define a norm on $c_{00}$, the linear space of all finitely supported sequences of scalars, take $X$ to be the completion, and the weakly null sequence will be the unit vector basis $(e_i)$ which will be a basis for
$X$. $(e_i)$ will have the property that the summing basis $(s_i)$ is equivalent to a block basis of each subsequence $(e_{n_i})$. The \textit{summing basis} is a basis for an isomorph of $c_0$ given by

$$\left\| \sum a_is_i \right\| = \sup \left| \sum_{1}^{n} a_i \right| .$$

Since $\left\| \sum_{1}^{n} s_i \right\| = n$ but $\left\| \sum_{1}^{n} (-1)^is_i \right\| = 1$ it is conditional.

The Maurey-Rosenthal example also illustrates the use of a coding in constructing a Banach space. A general procedure for constructing a norm on $c_{00}$ so that in the completion $(e_i)$ is a monotone normalized basis is to choose a certain set of functionals $\mathcal{F} \subseteq [-1,1]^\mathbb{N}$ with $e_i \in \mathcal{F}$ for all $i$ and if $f \in \mathcal{F}$ and $n \in \mathbb{N}$ then $P_nf = f|_{[1,n]} \in \mathcal{F}$. The norm is then given for $x \in c_{00}$ by

$$\|x\|_{\mathcal{F}} \equiv \sup_{f \in \mathcal{F}} |\langle f, x \rangle| = \sup_{f \in \mathcal{F}} \left| \sum_{1}^{\infty} f(i)x(i) \right|$$

If $X = (c_{00}, \| \cdot \|_{\mathcal{F}})$ then $(e_n)$ is weakly null if $f(e_n) \to 0$ for all $f \in \overline{\mathcal{F}}$ (pointwise closure of $\mathcal{F}$) since naturally $X \subseteq C(\overline{\mathcal{F}})$. $\overline{\mathcal{F}}$ is actually the $\omega^*$-closure of $\mathcal{F}$ in $B_{X^*}$.

Returning to the M-R example we let $\varepsilon_i \downarrow 0$ rapidly and choose integers $1 = m_1 < m_2 < \cdots$ so that if
$i < j$ then $\sqrt[m_i]{m_j} < \varepsilon_j$. For sets of integers $E, F \subseteq \mathbb{N}$ we write $E < F$ if $\max E < \min F$. Set

$$\vec{F} = \left\{ (F_i)^n_1 : n \in \mathbb{N}, \ F_i \subseteq \mathbb{N}, \ F_1 < F_2 < \cdots < F_n \right\}$$

and let $\phi : \vec{F} \to (m_i)^\infty$ be 1–1 ($\phi$ is called a coding).

Set (for $E \subseteq \mathbb{N}$, $|E|$ denotes the cardinality of $E$)

$$\mathcal{F} = \left\{ f = \sum_{i=1}^{n} \frac{1_{E_i}}{\sqrt{|E_i|}} : n \in \mathbb{N}, \ (E_i)^n_1 \in \vec{F}, \ |E_1| = 1 \right\}$$

and $|E_{i+1}| = \phi(E_1, \ldots, E_i)$.

Note that if $|E| = m_i, |F| = m_j$ with $i < j$ then

$$\left\langle \frac{1_E}{\sqrt{|E|}}, \frac{1_F}{\sqrt{|F|}} \right\rangle \leq \frac{m_i}{\sqrt{m_i m_j}} = \frac{\sqrt{m_i}}{\sqrt{m_j}} < \varepsilon_j.$$

Then $X_{MR} = (c_{00}, \| \cdot \|_{\mathcal{F}})$.

From this setup it is not hard to show that given a subsequence $N$ of $\mathbb{N}$ if $E_1 < E_2 < \cdots$ are subsets of $N$ with $|E_1| = 1$ and $|E_{i+1}| = \phi(E_1, \ldots, E_i)$ then $x_i = \frac{1_{E_i}}{\sqrt{|E_i|}}$ is equivalent to $(s_i)$. For all $m$, $(a_i)^m_1 \subseteq \mathbb{R},$

$$\sup_{n \leq m} \left| \sum_{1}^{n} a_i \right| \leq \left\| \sum_{1}^{m} a_i x_i \right\| \leq 4 \sup_{n \leq m} \left| \sum_{1}^{n} a_i \right|.$$
The left hand inequality comes from taking
\[ f_n = \sum_{1}^{n} \frac{1_{E_i}}{\sqrt{|E_i|}} \in \mathcal{F} \text{ for } n \leq m. \]

For the right hand inequality we use that if \( f \in \mathcal{F} \) then \( f \) agrees with \( f_n \), as defined above, for some \( n \), then the next term could act partially on \( a_{n+1}x_{n+1} \) but the ensuing terms have different associated \( m_j \)'s and so are nearly orthogonal to \( \sum_{n+1}^{m} a_i x_i \). More precisely let
\[
f = \sum_{1}^{k} \frac{1_{F_i}}{\sqrt{|F_i|}} \in \mathcal{F} \quad \text{and} \quad i_0 = \max\{i : F_i = E_i\}
\]
\[
\left| \left\langle f, \sum_{1}^{m} a_i \frac{1_{E_i}}{\sqrt{|E_i|}} \right\rangle \right|
\]
\[
\leq \left| \sum_{1}^{i_0} a_i \right| + \left| \left\langle \frac{1_{F_{i_0+1}}}{\sqrt{|F_{i_0+1}|}}, \sum_{1}^{m} a_i \frac{1_{E_i}}{\sqrt{|E_i|}} \right\rangle \right|
\]
\[
+ \sum_{j=i_0+1}^{k} \sum_{i=1}^{m} |a_i| \left| \left\langle \frac{1_{E_j}}{\sqrt{|F_j|}}, \frac{1_{E_i}}{\sqrt{|E_i|}} \right\rangle \right|.
\]
If \( |F_{i_0+1}| = m_s \) the second term is
\[
\leq \max_i |a_i| \left( 1 + s\varepsilon_s + \sum_{s+1}^{\infty} \varepsilon_i \right)
\]
and the third term is
\[
\leq \max_i |a_i| \left( \sum_j \left( j \varepsilon_j + \sum_{t=j+1}^{\infty} \varepsilon_t \right) \right)
\]

Equation (1.1) follows if \( \varepsilon_i \)'s are sufficiently small.

We should also mention a recent result of Johnson, Maurey and Schechtman [JMS]. This is much more difficult.

**Theorem 1.8.** There exists a normalized weakly null sequence \((f_i)\) in \(L_1\) so that for each subsequence \((f_{n_i})\) and \(\varepsilon > 0\) the Haar basis for \(L_1\) is \(1 + \varepsilon\)-equivalent to a block basis of \((f_{n_i})\).

Since the Haar basis for \(L_1\) is conditional this proves that no subsequence of \((f_i)\) is unconditional.

We will focus for a while on the problem of finding a “nice” subspace of a given \(X\). For our future considerations of “nice” by passing to a basic sequence in \(X\) we may assume \(X\) has a basis, \((e_i)\). In fact we could think of \(X\) as \((c_{00}, \| \cdot \|_{B_{X^*}})\).

We need to search among all subspaces of \(X\) to find a “nice” one. The easiest search is among all subspaces \([(e_{n_i})]\) and this leads us naturally to Ramsey theory. As we shall see this will prove productive but ultimately we will have to search among all block bases for, say,
an unconditional basic sequence. The Ramsey theorem that would yield a positive result fails however, of course. But partial results ensue.
References


19


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