Palm Calculus

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On the background there is a measurable space $(\Omega, \mathcal{F})$ equipped with a measurable flow $\theta : (\Omega \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}(\mathbb{R})) \longmapsto (\Omega, \mathcal{F})$ satisfying

(a) θ_t is bijective for all $t \in \mathbb{R}$.

(b) $\theta_t \circ \theta_t = \theta_{s+t}$ for all $s, t \in \mathbb{R}$. In particular, θ_0 is the identity and $\theta_t^{-1} = \theta_{-t}$.

We further assume that a θ–flow invariant probability measure $\mathbb{P}$ is defined on the former measurable space, i.e., $\mathbb{P}[\theta_t^{-1} C] = \mathbb{P}[C]$ for every $t \in \mathbb{R}$ and $C \in \mathcal{F}$. Let A be a random measure on $\mathcal{B}(\mathbb{R})$ and let the intensitty measure of A be denoted by ν_A. Also, let us assume that

(i) $A(B) \circ \theta_t = A(B + t) \circ \theta_0$. In other words, A is stationary with respect to flow θ,

(ii) $0 < \mathbb{E}\left[A\left((0, 1]\right)\right] < \infty$. We call the quatity in–between the inequality intensity of A, which we denote by λ_A; in the curent description, we say that A has finite intensity.

By a monotone class argument we can deduce from *(i)* that for any positive $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$–measurable function f,

$$\int f(s, \theta_t \omega) A(ds, \theta_t \omega) = \int f(s - t, \theta_t \omega) A(ds, \omega) \tag{1}$$

Now we use A to define a random measure $\mathcal{A}$ on $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$ by setting

$$\mathcal{A}(L \times C, \omega) = \int_{\mathbb{R}} \mathbf{1}_C(s) \mathbf{1}_L(\theta_s \omega) A(ds, \omega) \tag{2}$$

Using shifting equation (1) on defining equation (2), we obtain

$$\mathcal{A}(L \times C, \theta_t\omega) = \mathcal{A}(L \times (C+t), \omega)$$

In the present framework, we define the *Cambell measure* $\nu_{A,\theta}$ associated to the pair (A, θ) by setting

$$\begin{aligned} \nu_{A,\theta}(L \times C) &= \mathbb{E}\left[\mathcal{A}(L \times C)\right] \qquad (3) \\ &= \mathbb{E}\left[\int_{\mathbb{R}} \mathbf{1}_C(s)\mathbf{1}_L(\theta_s)A(ds)\right] \end{aligned}$$

Clearly, for any bounded Borel subset C of $\mathbb{R}$ we have that $\nu_{A,\theta}(L \times C) \leq \nu_A(C) < \infty$. On the other hand, the θ–invariance of $\mathbb{P}$ tells us that

$$\mathbb{E}\left[\mathcal{A}(L \times (C+t))\right] = \mathbb{E}[\mathcal{A}(L \times C) \circ \theta_t] = \mathbb{E}[\mathcal{A}(L \times C)]$$

This and property *(ii)* mean that for $L \in \mathcal{F}$ fixed, $\nu_{A,\theta}$ is a σ–finite translation invariant measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, thus we can rewrite equation (3) as

$$\begin{aligned} \nu_{A,\theta}(L \times C) &= \nu_{A,\theta}(L \times (0,1])|C| \\ &= \lambda_A|C|\left[\frac{1}{\lambda_A}\nu_{A,\theta}(L \times (0,1])\right] \qquad (4) \end{aligned}$$

where $|C|$ stands for the Lebesgue measure in $\mathbb{R}$ of C. The expression inside the parenthesis on (4) defines a probability measure $\mathbb{P}_A$ on $(\Omega, \mathcal{F})$, which receives the special name of *Palm Probability* of A. Denoting by $\mathbb{E}_A$ the expectation with respect Palm probability $\mathbb{P}_A$ we have that for any positive $\mathcal{F}$–measurable function F,

$$\lambda_A\mathbb{E}_A[F] = \mathbb{E}\left[\int_{(0,1]} F \circ \theta_s A(ds)\right] \qquad (5)$$

Also, notice that whenever $C \in \mathcal{B}(\mathbb{R})$ with $0 < |C| < \infty$, we have that

$$\mathbb{P}_A[L] = \frac{1}{\lambda_A|C|}\mathbb{E}\left[\int_{\mathbb{R}} \mathbf{1}_C(s)\mathbf{1}_L \circ \theta_s\, A(ds)\right]$$

Starting with measurable functions of the form $v(\omega, t) = \mathbf{1}_L(\omega)\mathbf{1}_C(t)$ with $C \subset \mathbb{R}$ bounded, a simple application of Fubini's theorem and then using monotone class arguments to deal with non–negative $(\mathcal{F}\otimes\mathcal{B}(\mathbb{R}))$–measurable functions we obtain the following result:

Theorem 0.1. *(Mecke's formula) Under the foremost assumptions*

$$\lambda_A \int_{\mathbb{R}} \int_{\Omega} v(\omega, s)\mathbb{P}_A(d\omega)\, ds = \int_{\Omega} \int_{\mathbb{R}} v(\theta_s\omega, s)A(ds, \omega)\mathbb{P}(dw)$$

It is easy to remember the last equation in terms of expectations as:

$$\lambda_A \mathbb{E}_A \left[\int_{\mathbb{R}} v(\theta_0, s)\, ds \right] = \mathbb{E} \left[\int_{\mathbb{R}} v(\theta_s, s)\, A(ds) \right] \tag{6}$$

In particular, for the case in which A is a stationary point process we can obtain the inversion formula of Ryll–Nardzewski and Slivnyak. We set

$$v(\omega, t) = \mathbf{1}_{[0,T_1)}(-t) f(\theta_{-t}\omega)$$

where f is a nonnegative $\mathcal{F}$–measurable function. Using the symmetry of Lebesgue measure one hand and noticing that

$$\mathbf{1}_{(-T_1 \circ \theta_{T_n}, 0]}(T_n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

we get

$$\mathbb{E}[f] = \lambda_A \mathbb{E}_A \left[\int_0^{T_1} f \circ \theta_s\, ds \right]$$

One consequence of Mecke's formula is

Theorem 0.2. *(Integrals of Random Product Measures) Suppose M and N are random measures on $\mathbb{R}$ that are stationary with respect to the flow θ and have intensities λ_M and λ_N respectively, both of which are positive and finite. Then for any non–negative $(\mathbb{R}^2 \times \Omega, \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{F})$–measurable function g we have*

$$\mathbb{E} \left[\int_{\mathbb{R}^2} g(t, s, \theta_t) N(ds) M(dt) \right] = \lambda_M \mathbb{E}_M \left[\int_{\mathbb{R}^2} g(t, t+s, \theta_0) N(ds)\, dt \right] \tag{7}$$

Proof. Let us set $v(t, \omega) = \int_{\mathbb{R}} g(t, t+s, \omega) N(ds, \omega)$. Then by Fubini's theorem, the left hand side of (7) takes the form

$$\lambda_M \mathbb{E}_M \left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(t, t+s, \theta_0) N(ds) \right) dt \right] = \lambda_M \mathbb{E}_M \left[\int_{\mathbb{R}} v(\theta_0, t)\, dt \right]$$

Applying Mecke's formula (6) and the shifting formula (1) we obtain the desired result. □

We use the last theorem to prove an important exchange formula:

Theorem 0.3. *(Neveu's Exchange Formula) Let M and N as before. Then for any positive $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F})$–measurable function f*

$$\lambda_M \mathbb{E}_M \left[\int_{\mathbb{R}} f(t, \theta_t) N(dt) \right] = \lambda_N \mathbb{E}_N \left[\int_{\mathbb{R}} f(-t, \theta_0) M(dt) \right] \tag{8}$$

Proof. Using (5), the shifting equation (1) and Fubini's theorem we get that

$$\lambda_M \mathbb{E}_M \left[\int_{\mathbb{R}} f(t, \theta_t) N(dt) \right] = \mathbb{E} \left[\int_{\mathbb{R}^2} \mathbf{1}_{(0,1]}(s) f(t-s, \theta_t) M(ds) N(dt) \right]$$

Further, by setting $g(t, s, \omega) = \mathbf{1}_{(0,1]}(s) f(t-s, \omega)$ and then applying the formula for integrals of product measures (7), together with Fubini's theorem we get the desired result. □

By specializing Neveu's formula (8) to the case in which both M and N are stationary point processes we can get the well known Neveu's formula in discrete case. We only have to set

$$f(t, \theta_0) = \mathbf{1}_{[0, T_1' \circ \theta_0)}(-t) h(\theta_{-t} \circ \theta_0)$$

where T_n' is the n–th time of N. Then notice that

$$\mathbf{1}_{(-T_1' \circ \theta_{T_n'}, 0]}(T_n') = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then equation (8) reads as

$$\lambda_M \mathbb{E}_M[h] = \lambda_N \mathbb{E}_N \left[\int_{\mathbb{R}} \mathbf{1}_{[0, T_1')}(t) h(\theta_t) M(dt) \right]$$

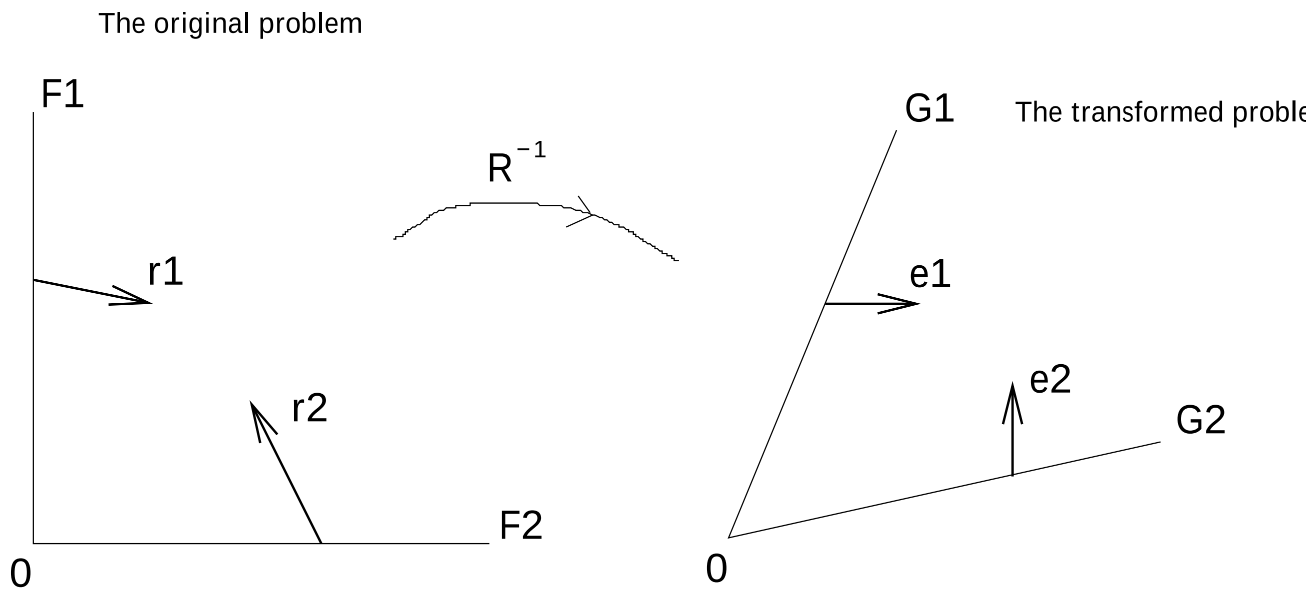
We say that a measurable random measure N is adapted to a filtration $\mathcal{F}_t$ with $t \in \mathbb{R}$, if $N(C)$ is $\mathcal{F}_t$–adapted for any $C \in \mathcal{B}((-\infty, t])$. The *internal history* of N is defined as $\mathcal{F}_t^N = \sigma(\{N(C) : C \in \mathcal{B}((-\infty, t]))$. A process Z_t is adapted to the flow θ if $Z_t = Z_0 \circ \theta_t$. Assuming N is a point process stationary with respect to the flow θ and Z as before, then, we say that $\{Z_{\theta_{T_n}}\}$ is a sequence of *marks* of N and that (N, Z) is a *marked point process.* The *intenal history* of (N, Z) is defined as $\mathcal{F}_t^{(N,Z)} = \mathcal{F}_t^N \wedge \mathcal{F}_t^Z$. If $\mathcal{F}_t^{(N,Z)} \subset \mathcal{F}_t$ then we say that (N, Z) is adapted to $\mathcal{F}_t$.
A process Z is *progressivele* measurable if for any t, the stopped process Z^t

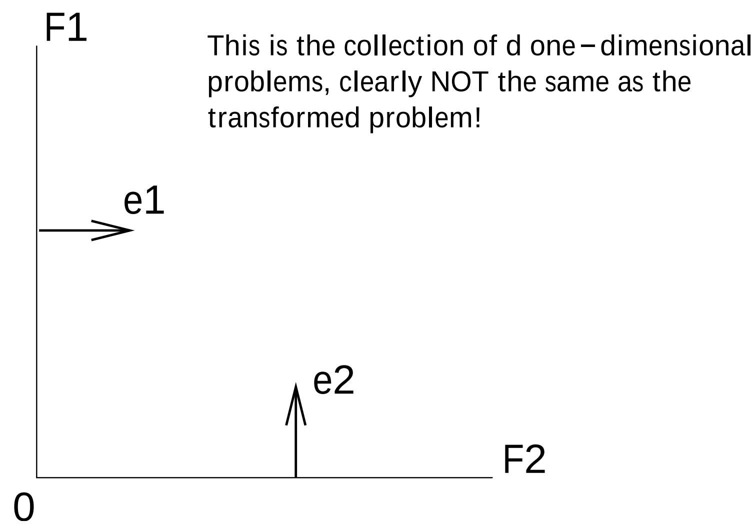
defined by $Z^t(\omega, s) = Z(\omega, s \vee t)$, is $\mathcal{F} \otimes \mathcal{B}((\infty, t])$–measurable. It is easy to check that progressively measurable processes are adapted and that adapted processes that are either cádlág or cágl'ag are progressively measurable.
The σ–algebra $\mathcal{P}(\mathcal{F}_t) = \sigma((a, b] \times A : a \leq b \quad A \in \mathcal{F}_a)$ defined on $\mathbb{R} \times \Omega$ is called the $\mathcal{F}_t$–*predictable* σ–algebra. If the process Z_t is such that $(t, \omega) \mapsto Z(t, \omega)$ is $\mathcal{P}(\mathcal{F}_t)$–measurable, then it is called $\mathcal{F}_t$–predictable. It can be shown that the σ–algebra generated by the adapted cáglád processes coincides with the predictable σ–algebra, see Last & Brandt, p. 45.
The filtration $\mathcal{F}_t$ is said to be *compatible* with the flow θ_t if $\theta_t \mathcal{F}_s = \mathcal{F}_{s-t}$ for all $s, t \in \mathbb{R}$. Moreover, if any predictable process Z_t has the form

$$Z(t, \omega) = v(t, \theta_t \omega) \tag{9}$$

where v is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$–measurable, and for any t, v_t is $\mathcal{F}_{-0}$–measurable (where $\mathcal{F}_{0-} = \bigwedge_{s<0} \mathcal{F}_s$), then we say that $\mathcal{F}_t$ has *predictable structure* adapted to the flow θ. The following result, see Baccelli & Brémaud p. 47, justifies the introduction of the last two definitions.

Lemma 0.4. *Let N be a random measure stationary for θ and Z_t a process compatible with the flow. Then, the filtration $\mathcal{F}_t^{N,Z}$ has a predictable structure. Furthermore, if N is a point process then, any process as in (9) with v as before, is predictable.*


The original problem
F1
r1
r2
F2
0
R^{-1}
G1
The transformed proble
e1
e2
G2
0


F1
This is the collection of d one−dimensional problems, clearly NOT the same as the transformed problem!
e1
e2
F2
0

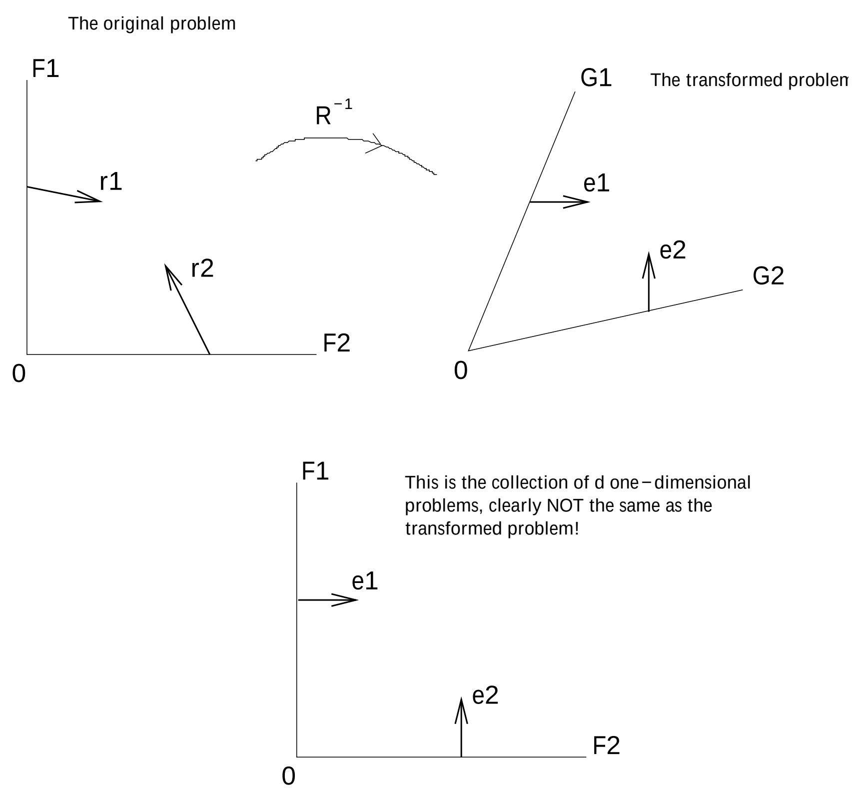


Figure 1: Appears as Fig. Caption under the Drawing