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SUPPLEMENTARY MATERIALS: Theorems on the Stokesian Hydrodynamics of a Rigid Filament in the Limit of Vanishing Radius*

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Here we provide details for the proofs of Lemmas 4.1–4.3. Numbers with the label S denote equations and references within this supplement; numbers without this label refer to the main article.

Proof of Lemma 4.1. We first consider the resistance problem, in which the quantities $(V, \Omega, u^\infty, p^\infty)$ are given, and $(F^{\rm ext}, T^{\rm ext}, u^+, p^+)$ are sought. In view of (2.1)–(2.4), we introduce the shifted fields $w^+ = u^+ - u^\infty$ and $q^+ = p^+ - p^\infty$, and consider a decomposition $w^+ = \widetilde{w}^+ - \widehat{w}^+$ and $q^+ = \widetilde{q}^+ - \widehat{q}^+$, where $(\widetilde{w}^+, \widetilde{q}^+)$ and $(\widehat{w}^+, \widehat{q}^+)$ satisfy

$$\text{(SM0.1)}\qquad \begin{array}{lll} \Delta\widetilde{w}^+ = \nabla\widetilde{q}^+, & \Delta\widehat{w}^+ = \nabla\widehat{q}^+, & x \in D^+ \\ \nabla\cdot\widetilde{w}^+ = 0, & \nabla\cdot\widehat{w}^+ = 0, & x \in D^+ \\ \widetilde{w}^+ = V + \Omega\times(x-c), & \widehat{w}^+ = u^\infty, & x \in \Gamma \\ \widetilde{w}^+, \widetilde{q}^+ \to 0, 0, & \widehat{w}^+, \widehat{q}^+ \to 0, 0, & |x| \to \infty. \end{array}$$

Since Γ is closed, bounded and Lyapunov, and the boundary data $V + \Omega \times (x - c)$ and u^∞ are continuous, it follows from classic results of potential theory for the Stokes equations [SM2, SM4, SM5, SM6] that each of the two systems in (SM0.1) has a unique solution. The fields $(\widetilde{w}^+, \widetilde{q}^+)$ and $(\widehat{w}^+, \widehat{q}^+)$ are smooth in D^+, and $\widetilde{w}^+$ and $\widehat{w}^+$ are continuous up to Γ. Due to the rigid-body form of the boundary data for $(\widetilde{w}^+, \widetilde{q}^+)$, the associated traction field $\widetilde{h}^+ = \widetilde{\sigma}^+\nu$ is guaranteed to be continuous up to Γ [SM3, SM4, SM5], and from integral moments of $\widetilde{h}^+$ we obtain well-defined resultant loads $(\widetilde{F}, \widetilde{T})$. Since the boundary data for $(\widehat{w}^+, \widehat{q}^+)$ is not only continuous, but also in the Sobolev space $H^{1/2}$ on Γ, the associated traction field $\widehat{h}^+ = \widehat{\sigma}^+\nu$ is guaranteed to be in the dual space $H^{-1/2}$ [SM1], which implies that integral moments of $\widehat{h}^+$ and hence resultant loads $(\widehat{F}, \widehat{T})$ are also well-defined. In view of (2.7) and (2.8), and the fact that the far-field flow (u^∞, p^∞) generates zero loads $(F^\infty, T^\infty) = (0, 0)$ on any closed boundary, the external loads are well-defined and given by $(F^{\rm ext}, T^{\rm ext}) = (\widehat{F} - \widetilde{F}, \widehat{T} - \widetilde{T})$. Hence, given $(V, \Omega, u^\infty, p^\infty)$, there is a unique flow $(u^+, p^+) = (w^+ + u^\infty, q^+ + p^\infty)$, which is k-times continuously differentiable in D^+, and unique external loads $(F^{\rm ext}, T^{\rm ext})$ for the resistance problem. Moreover, the assumption that $\sigma[u^+, p^+]$ is continuous up to Γ implies that $\sigma[w^+, q^+]$ must also be continuous up to Γ, which will be exploited below.

We next consider the mobility problem, in which $(F^{\rm ext}, T^{\rm ext}, u^\infty, p^\infty)$ are given, and (V, Ω, u^+, p^+) are sought. In this case, just as before, the second system in (SM0.1) has a unique solution $(\widehat{w}^+, \widehat{q}^+)$, with well-defined resultant loads $(\widehat{F}, \widehat{T})$. For the first system in (SM0.1), the loads $(\widetilde{F}, \widetilde{T}) = (\widehat{F} - F^{\rm ext}, \widehat{T} - T^{\rm ext})$ are now given, and the flow fields $(\widetilde{w}^+, \widetilde{q}^+)$ and body velocities (V, Ω) are unknown. Due to the rigid-body form of the given and unknown

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data on Γ, the first system in (SM0.1) has a unique solution for $(\widetilde{w}^+, \widetilde{q}^+)$ and (V, Ω) [SM3]. As before, the fields $(\widetilde{w}^+, \widetilde{q}^+)$ and $(\widehat{w}^+, \widehat{q}^+)$ are smooth in D^+, with $\widetilde{w}^+$ and $\widehat{w}^+$ continuous up to Γ. Hence, given $(F^{\rm ext}, T^{\rm ext}, u^\infty, p^\infty)$, there is a unique flow $(u^+, p^+) = (w^+ + u^\infty, q^+ + p^\infty)$, which is k-times continuously differentiable in D^+, and unique body velocities (V, Ω) for the mobility problem. The assumption that $\sigma[u^+, p^+]$ is continuous up to Γ again implies that $\sigma[w^+, q^+]$ must also be continuous up to Γ.

As a companion to the exterior problem, regardless of its type (resistance or mobility), we may consider an auxiliary interior problem for fields (w^-, q^-), namely

$$\begin{aligned} &\Delta w^- = \nabla q^-, && x \in D^- \\ &\nabla \cdot w^- = 0, && x \in D^- \\ &w^- = V + \Omega \times (x - c) - u^\infty, && x \in \Gamma. \end{aligned} \tag{SM0.2}$$

Since the boundary data is continuous and satisfies $\int_\Gamma w^- \cdot \nu \, dA_x = 0$, this interior problem has a solution, where w^- is unique, and q^- is unique up to an additive constant C [SM4, SM5, SM6]. By inspection, the general solution in this case is $w^- = V + \Omega \times (x - c) - u^\infty$ and $q^- = -p^\infty + C$. We note that (w^-, q^-) are k-times continuously differentiable in D^-, with w^- and $\sigma[w^-, q^-]$ continuous up to Γ.

Well-known results for the Stokes equations, based on the fundamental solution of the equations and the divergence theorem, imply that the solution of the systems in (SM0.1) and hence the overall exterior fields (w^+, q^+) possess a natural representation involving both the single- and double-layer Stokes potentials [SM4, SM5, SM6, SM7]. Sufficient conditions for this representation are that (w^+, q^+) be twice continuously differentiable in D^+, with w^+ and $\sigma[w^+, q^+]$ continuous up to Γ, and decaying at infinity. Similarly, the interior fields (w^-, q^-) also possess such a representation, and for this it is sufficient that (w^-, q^-) be twice continuously differentiable in D^-, with w^- and $\sigma[w^-, q^-]$ continuous up to Γ. The assumed regularity up to the boundary implies that, in both the exterior and interior cases, the representations hold with continuous densities. By combining the two representations, and using the fact that $w^+ = w^-$ on Γ, the double-layer terms can be eliminated to obtain purely single-layer representations for (w^+, q^+) and (w^-, q^-). Specifically, we find

$$w^+(x) = U[\Gamma, \psi](x), \qquad q^+(x) = P[\Gamma, \psi](x), \qquad x \in D^+ \tag{SM0.3}$$

with the continuous density $\psi = \frac{1}{8\pi}(\sigma[w^-, q^-] - \sigma[w^+, q^+])\nu$ on Γ. This representation holds for any choice of the pressure constant C in the interior flow (the single-layer velocity potential has a one-dimensional nullspace), and for convenience we take $C = 0$. By direct computation we have $\sigma[w^-, q^-] = -\sigma[u^\infty, p^\infty]$, and by construction we have $\sigma[w^+, q^+] = \sigma[u^+, p^+] - \sigma[u^\infty, p^\infty]$, which implies $\psi = -\frac{1}{8\pi} h^+$. Hence the expressions in (4.5) and (4.6) are established. The boundary integral equation in (4.7) follows from $(4.5)_1$, together with the boundary conditions in $(2.2)_1$ and (2.4), and the limit relation in (4.3).

Proof of Lemma 4.2. In view of (4.2) and (4.8) we have

$$\mathcal{E}(x) = \mathcal{I}(x) + \mathcal{A}(x), \tag{SM0.4}$$

where

$$\mathcal{I}_{ij}(x) = \frac{1}{8\pi} \int_{\Gamma_{r,\rm str}} \frac{\delta_{ij}}{|x - y|} \, dA_y, \quad \mathcal{A}_{ij}(x) = \frac{1}{8\pi} \int_{\Gamma_{r,\rm str}} \frac{(x - y)_i (x - y)_j}{|x - y|^3} \, dA_y. \tag{SM0.5}$$

By properties of weakly-singular integrals, we note that the components $\mathcal{I}_{ij}(x)$ and $\mathcal{A}_{ij}(x)$ are well-defined and continuous functions of $x \in \Gamma_{r,\mathrm{str}}$ for each $r > 0$, and here we seek to characterize their limits as $r \to 0^+$.

To begin, we consider the first term in (SM0.4). Let $x \in \Gamma_{r,\mathrm{str}}$ be given, and without loss of generality assume $\theta_x = 0$, so that $x = (r, 0, s_x)$ with $-L < s_x < L$. Using the local coordinate representations $y = (r\cos\theta_y, r\sin\theta_y, s_y)$ and $dA_y = r d\theta_y ds_y$, we have

$$\mathcal{I}_{ij}(x) = \frac{1}{8\pi}\int_{-L}^{L}\int_{0}^{2\pi} \frac{\delta_{ij}}{\sqrt{r^2(2-2\cos\theta_y) + (s_x - s_y)^2}}\, r d\theta_y ds_y. \tag{SM0.6}$$

For convenience, let $\phi = \theta_y$ and $\sigma = s_y - s_x$, and define $\sigma_{x,+} = L - s_x > 0$ and $\sigma_{x,-} = -L - s_x < 0$, and note that $\sigma_{x,+} = 0$ and $\sigma_{x,-} = 0$ only at the endpoints $s_x = L$ and $s_x = -L$. In terms of ϕ and σ we have

$$\mathcal{I}_{ij}(x) = \frac{1}{8\pi}\int_{\sigma_{x,-}}^{\sigma_{x,+}}\int_{0}^{2\pi} \frac{\delta_{ij}}{\sqrt{r^2(2-2\cos\phi) + \sigma^2}}\, r d\phi d\sigma. \tag{SM0.7}$$

Introducing the additional change of variable $\sigma = r\eta$, the integral can be written in the form

$$\mathcal{I}_{ij}(x) = \frac{r\delta_{ij}}{8\pi}\int_{0}^{\frac{1}{r}\sigma_{x,+}} f(\eta)\, d\eta + \frac{r\delta_{ij}}{8\pi}\int_{0}^{\frac{1}{r}|\sigma_{x,-}|} f(\eta)\, d\eta, \tag{SM0.8}$$

where $f(\eta)$ is a positive, even function defined by

$$f(\eta) = \int_{0}^{2\pi} \frac{1}{\sqrt{(2-2\cos\phi) + \eta^2}}\, d\phi, \qquad \eta \neq 0. \tag{SM0.9}$$

It will be useful to characterize the behavior of $f(\eta)$ in the limits $\eta \to 0^+$ and $\eta \to \infty$. Noting that the integrand and interval are symmetric about $\phi = \pi$, and using the change of variable $\phi = 2\xi$, we get

$$f(\eta) = \int_{0}^{\pi} \frac{2}{\sqrt{(2-2\cos\phi) + \eta^2}}\, d\phi = \int_{0}^{\pi/2} \frac{4}{\sqrt{4\sin^2\xi + \eta^2}}\, d\xi. \tag{SM0.10}$$

To characterize the limit as $\eta \to 0^+$, let $\widehat{\epsilon} \in (0, \frac{\pi}{2})$ be fixed, and consider the decomposition

$$f(\eta) = \int_{0}^{\widehat{\epsilon}} \frac{4}{\sqrt{4\sin^2\xi + \eta^2}}\, d\xi + \int_{\widehat{\epsilon}}^{\pi/2} \frac{4}{\sqrt{4\sin^2\xi + \eta^2}}\, d\xi. \tag{SM0.11}$$

For the first term above, we introduce the change of variables $t = 2\sin\xi$ so that $dt = \sqrt{4-t^2}\, d\xi$. Noting that $\sqrt{4-\epsilon^2} \le \sqrt{4-t^2} \le 2$ for all $0 \le t \le \epsilon$, where $\epsilon = 2\sin\widehat{\epsilon}$, and noting that $\int_0^{\epsilon}(t^2+\eta^2)^{-1/2}\, dt$ can be evaluated in closed form, we obtain

$$\frac{4}{2}\ln\Big(\frac{\epsilon + \sqrt{\epsilon^2+\eta^2}}{\eta}\Big) \le \int_{0}^{\widehat{\epsilon}} \frac{4}{\sqrt{4\sin^2\xi + \eta^2}}\, d\xi \le \frac{4}{\sqrt{4-\epsilon^2}}\ln\Big(\frac{\epsilon + \sqrt{\epsilon^2+\eta^2}}{\eta}\Big). \tag{SM0.12}$$

The above inequality shows that the first term in (SM0.11) grows logarithmically, whereas the second term remains bounded, and hence $f(\eta) \to \infty$ as $\eta \to 0^+$. Indeed, combining (SM0.11) and (SM0.12), we find that $2 \le \lim_{\eta\to 0^+}[f(\eta)/\ln(\frac{1}{\eta})] \le \frac{4}{\sqrt{4-\epsilon^2}}$. From this we deduce, by the arbitrariness of ϵ, that

$$\lim_{\eta\to 0^+} \frac{f(\eta)}{\ln(\frac{1}{\eta})} = 2. \tag{SM0.13}$$

To characterize the limit as $\eta \to \infty$, let $\epsilon > 0$ be fixed, and note that $1 \le 1 + 4\sin^2(\xi)/\eta^2 \le 1+\epsilon^2$ for all $\eta \ge 2/\epsilon$. In view of (SM0.10) we obtain

$$\frac{2\pi}{\eta\sqrt{1+\epsilon^2}} \le \int_0^{\pi/2} \frac{4}{\sqrt{4\sin^2\xi + \eta^2}}\, d\xi \le \frac{2\pi}{\eta}. \tag{SM0.14}$$

The above shows that $f(\eta) \to 0$ as $\eta \to \infty$; specifically, by the arbitrariness of ϵ, we obtain

$$\lim_{\eta\to\infty} \frac{f(\eta)}{\frac{1}{\eta}} = 2\pi. \tag{SM0.15}$$

Next, given any $\epsilon > 0$, the limits in (SM0.13) and (SM0.15) imply the following inequalities for some intervals $(0, a_\epsilon)$ and (b_ϵ, ∞), where $a_\epsilon < 1$ and $b_\epsilon > 1$,

$$2-\epsilon \le \frac{f(\eta)}{\ln(\frac{1}{\eta})} \le 2+\epsilon, \quad \eta \in (0, a_\epsilon), \tag{SM0.16}$$

$$2\pi-\epsilon \le \frac{f(\eta)}{\frac{1}{\eta}} \le 2\pi+\epsilon, \quad \eta \in (b_\epsilon, \infty). \tag{SM0.17}$$

Multiplying the first inequality by $\ln(\frac{1}{\eta})$, and integrating over $(0, a_\epsilon)$, we get

$$(2-\epsilon)a_\epsilon(1-\ln a_\epsilon) \le \int_0^{a_\epsilon} f(\eta)\, d\eta \le (2+\epsilon)a_\epsilon(1-\ln a_\epsilon). \tag{SM0.18}$$

Similarly, multiplying the second inequality by $\frac{1}{\eta}$, and integrating over (b_ϵ, c) for any $c > b_\epsilon$, we obtain

$$(2\pi-\epsilon)(\ln c - \ln b_\epsilon) \le \int_{b_\epsilon}^{c} f(\eta)\, d\eta \le (2\pi+\epsilon)(\ln c - \ln b_\epsilon). \tag{SM0.19}$$

Using (SM0.18) and (SM0.19) we can examine the behavior of $\mathcal{I}_{ij}(x)$ in the limit $r \to 0^+$ for any given $x \in \Gamma_{r,\text{str}}$ and $\epsilon > 0$. From (SM0.8) we have

$$\frac{\mathcal{I}_{ij}(x)}{r\ln(\frac{1}{r})} = \frac{\delta_{ij}}{8\pi\ln(\frac{1}{r})}\int_0^{\frac{1}{r}\sigma_{x,+}} f(\eta)\, d\eta + \frac{\delta_{ij}}{8\pi\ln(\frac{1}{r})}\int_0^{\frac{1}{r}|\sigma_{x,-}|} f(\eta)\, d\eta. \tag{SM0.20}$$

For the first integral in (SM0.20) we have the decomposition

$$\int_0^{\frac{1}{r}\sigma_{x,+}} f(\eta)\, d\eta = \int_0^{a_\epsilon} f(\eta)\, d\eta + \int_{a_\epsilon}^{b_\epsilon} f(\eta)\, d\eta + \int_{b_\epsilon}^{\frac{1}{r}\sigma_{x,+}} f(\eta)\, d\eta, \tag{SM0.21}$$

which holds for all $0 < r < \frac{1}{b_\epsilon}\min\{\sigma_{x,+}, |\sigma_{x,-}|\}$ so that $\frac{1}{r}\sigma_{x,+} > b_\epsilon$. Since the integrals over $(0, a_\epsilon)$ and (a_ϵ, b_ϵ) are fixed and independent of r, they will vanish in the following limit, and we obtain

$$\lim_{r\to 0^+} \frac{1}{\ln(\frac{1}{r})} \int_0^{\frac{1}{r}\sigma_{x,+}} f(\eta)\, d\eta = \lim_{r\to 0^+} \frac{1}{\ln(\frac{1}{r})} \int_{b_\epsilon}^{\frac{1}{r}\sigma_{x,+}} f(\eta)\, d\eta. \tag{SM0.22}$$

Using (SM0.19) with $c = \frac{1}{r}\sigma_{x,+}$, and noting that $\ln c = \ln(\frac{1}{r}) + \ln \sigma_{x,+}$, we obtain

$$2\pi - \epsilon \le \lim_{r\to 0^+} \frac{1}{\ln(\frac{1}{r})} \int_0^{\frac{1}{r}\sigma_{x,+}} f(\eta)\, d\eta \le 2\pi + \epsilon, \tag{SM0.23}$$

which by the arbitrariness of ϵ implies that the value of the limit is 2π. The same arguments can be used to examine the limit of the second integral in (SM0.20), and the value of the limit is also 2π. These results hold for any given $x \in \mathit{\Gamma}_{r,\mathrm{str}}$, so that $-L < s_x < L$ and hence $\sigma_{x,+} > 0$ and $|\sigma_{x,-}| > 0$. Results for the endpoints $s_x = \pm L$ can also be considered; in this case, either $\sigma_{x,+}$ or $|\sigma_{x,-}|$ is zero, and the right-hand side of (SM0.20) reduces to only one integral. Thus we have a pointwise limit result for the function $\mathcal{I}_{ij}(x)$, namely

$$\lim_{r\to 0^+} \frac{\mathcal{I}_{ij}(x)}{r\ln(\frac{1}{r})} = \begin{cases} \frac{1}{2}\delta_{ij}, & s_x \in (-L, L), \\ \\ \frac{1}{4}\delta_{ij}, & s_x = \pm L. \end{cases} \tag{SM0.24}$$

We next consider the second term in (SM0.4), and since $\mathcal{A}_{ij}(x) = \mathcal{A}_{ji}(x)$, there are six independent components to examine. Proceeding as before, let $x \in \mathit{\Gamma}_{r,\mathrm{str}}$ be given, and without loss of generality assume $\theta_x = 0$, so that $x = (r, 0, s_x)$ with $-L < s_x < L$. In local coordinates we have $y = (r\cos\theta_y, r\sin\theta_y, s_y)$, so that $(x-y)_1 = r(1-\cos\theta_y)$, $(x-y)_2 = -r\sin\theta_y$ and $(x-y)_3 = s_x - s_y$, and moreover $dA_y = r d\theta_y ds_y$. In view of (SM0.5), the component $\mathcal{A}_{11}(x)$ is given by

$$\mathcal{A}_{11}(x) = \frac{1}{8\pi} \int_{-L}^{L} \int_0^{2\pi} \frac{r^2(1-\cos\theta_y)^2}{[r^2(2-2\cos\theta_y) + (s_x - s_y)^2]^{3/2}}\, r d\theta_y ds_y. \tag{SM0.25}$$

Introducing $\phi = \theta_y$, $\sigma = s_y - s_x$ and $\sigma = r\eta$ as before, the integral can be written in the form

$$\mathcal{A}_{11}(x) = \frac{r}{8\pi} \int_0^{\frac{1}{r}\sigma_{x,+}} f_{11}(\eta)\, d\eta + \frac{r}{8\pi} \int_0^{\frac{1}{r}|\sigma_{x,-}|} f_{11}(\eta)\, d\eta, \tag{SM0.26}$$

where $f_{11}(\eta)$ is a positive, even function defined by

$$f_{11}(\eta) = \int_0^{2\pi} \frac{(1-\cos\phi)^2}{[(2-2\cos\phi) + \eta^2]^{3/2}}\, d\phi, \quad \eta \neq 0. \tag{SM0.27}$$

Straightforward arguments using the fact that $1 - \cos\phi \ge 0$ show that $0 < f_{11}(\eta) \le C$ for all $\eta \in (0, 1)$, and $0 < f_{11}(\eta) \le C/\eta^3$ for all $\eta \in [1, \infty)$. Here and throughout C denotes a positive constant whose value may change from one appearance to the next. From this we deduce that each of the integrals $\int_0^{\frac{1}{r}\sigma_{x,+}} f_{11}(\eta)\,d\eta$ and $\int_0^{\frac{1}{r}|\sigma_{x,-}|} f_{11}(\eta)\,d\eta$ is bounded uniformly in r since

$$\int_0^\infty f_{11}(\eta)\,d\eta \le C. \tag{SM0.28}$$

Thus we obtain a pointwise limit result for the function $\mathcal{A}_{11}(x)$, namely

$$\lim_{r\to 0^+} \frac{\mathcal{A}_{11}(x)}{r\ln(\frac{1}{r})} = \begin{cases} 0, & s_x \in (-L, L), \\ 0, & s_x = \pm L. \end{cases} \tag{SM0.29}$$

The above pointwise limit will similarly vanish for all components $\mathcal{A}_{ij}(x)$ except $\mathcal{A}_{33}(x)$. For $\mathcal{A}_{12}(x)$ and $\mathcal{A}_{23}(x)$, the integrals themselves vanish for each $r > 0$ due to the fact that the integrands are odd functions of θ_y about $\theta_y = \pi$. For $\mathcal{A}_{13}(x)$ we get

$$\mathcal{A}_{13}(x) = \frac{r}{8\pi}\int_0^{\frac{1}{r}|\sigma_{x,-}|} f_{13}(\eta)\,d\eta - \frac{r}{8\pi}\int_0^{\frac{1}{r}\sigma_{x,+}} f_{13}(\eta)\,d\eta, \tag{SM0.30}$$

where $f_{13}(\eta)$ is an odd function defined by

$$f_{13}(\eta) = \int_0^{2\pi} \frac{(1-\cos\phi)\eta}{[(2-2\cos\phi)+\eta^2]^{3/2}}\,d\phi, \quad \eta \ne 0. \tag{SM0.31}$$

Arguments using the fact that $1 - \cos\phi \ge 0$, and the inequality

$$\eta \le [(2-2\cos\phi)+\eta^2]^{1/2}, \tag{SM0.32}$$

show that $0 < f_{13}(\eta) \le C$ for all $\eta \in (0,1)$, and $0 < f_{13}(\eta) \le C/\eta^2$ for all $\eta \in [1,\infty)$, which similar to before implies

$$\int_0^\infty f_{13}(\eta)\,d\eta \le C. \tag{SM0.33}$$

The above bound then implies a vanishing limit as in (SM0.29). For $\mathcal{A}_{22}(x)$ we get

$$\mathcal{A}_{22}(x) = \frac{r}{8\pi}\int_0^{\frac{1}{r}\sigma_{x,+}} f_{22}(\eta)\,d\eta + \frac{r}{8\pi}\int_0^{\frac{1}{r}|\sigma_{x,-}|} f_{22}(\eta)\,d\eta, \tag{SM0.34}$$

where $f_{22}(\eta)$ is a positive, even function defined by

$$f_{22}(\eta) = \int_0^{2\pi} \frac{\sin^2\phi}{[(2-2\cos\phi)+\eta^2]^{3/2}}\,d\phi, \quad \eta \ne 0. \tag{SM0.35}$$

Arguments based on the fact that $1-\cos\phi \ge 0$ and $1+\cos\phi \ge 0$, and the inequality

$$\sin^2\phi = \frac{1}{2}(1+\cos\phi)(2-2\cos\phi) \le \frac{1}{2}(1+\cos\phi)[(2-2\cos\phi)+\eta^2], \tag{SM0.36}$$

show that $0 < f_{22}(\eta) \le f(\eta)$ for all $\eta \in (0,1)$, and $0 < f_{22}(\eta) \le C/\eta^3$ for all $\eta \in [1,\infty)$, where $f(\eta)$ is the function in (SM0.9). Hence, by the integrability of $f(\eta)$ on $(0,1)$ established in (SM0.18), we obtain

$$\int_0^\infty f_{22}(\eta)\, d\eta \le C, \tag{SM0.37}$$

which implies a vanishing limit as in (SM0.29). For the remaining component $\mathcal{A}_{33}(x)$ we get

$$\mathcal{A}_{33}(x) = \frac{r}{8\pi}\int_0^{\frac{1}{r}\sigma_{x,+}} f_{33}(\eta)\, d\eta + \frac{r}{8\pi}\int_0^{\frac{1}{r}|\sigma_{x,-}|} f_{33}(\eta)\, d\eta, \tag{SM0.38}$$

where $f_{33}(\eta)$ is a positive, even function defined by

$$f_{33}(\eta) = \int_0^{2\pi} \frac{\eta^2}{[(2-2\cos\phi)+\eta^2]^{3/2}}\, d\phi, \quad \eta \neq 0. \tag{SM0.39}$$

Similar to before, it will be useful to characterize the behavior of $f_{33}(\eta)$ in the limits $\eta \to 0^+$ and $\eta \to \infty$. Noting that the integrand and interval are symmetric about $\phi = \pi$, and using the change of variable $\phi = 2\xi$, we get

$$f_{33}(\eta) = \int_0^{\pi} \frac{2\eta^2}{[(2-2\cos\phi)+\eta^2]^{3/2}}\, d\phi = \int_0^{\pi/2} \frac{4\eta^2}{[4\sin^2\xi+\eta^2]^{3/2}}\, d\xi. \tag{SM0.40}$$

To characterize the limit as $\eta \to 0^+$, let $\widehat{\epsilon} \in (0, \frac{\pi}{2})$ be fixed, and consider the decomposition

$$f_{33}(\eta) = \int_0^{\widehat{\epsilon}} \frac{4\eta^2}{[4\sin^2\xi+\eta^2]^{3/2}}\, d\xi + \int_{\widehat{\epsilon}}^{\pi/2} \frac{4\eta^2}{[4\sin^2\xi+\eta^2]^{3/2}}\, d\xi. \tag{SM0.41}$$

For the first term above, we again introduce the change of variables $t = 2\sin\xi$ so that $dt = \sqrt{4-t^2}\, d\xi$. Noting that $\sqrt{4-\epsilon^2} \le \sqrt{4-t^2} \le 2$ for all $0 \le t \le \epsilon$, where $\epsilon = 2\sin\widehat{\epsilon}$, and noting that $\int_0^\epsilon \eta^2(t^2+\eta^2)^{-3/2}\, dt$ can be evaluated in closed form, we obtain

$$\frac{4}{2}\frac{\epsilon}{\sqrt{\epsilon^2+\eta^2}} \le \int_0^{\widehat{\epsilon}} \frac{4\eta^2}{[4\sin^2\xi+\eta^2]^{3/2}}\, d\xi \le \frac{4}{\sqrt{4-\epsilon^2}}\frac{\epsilon}{\sqrt{\epsilon^2+\eta^2}}. \tag{SM0.42}$$

The above inequality shows that the first term in (SM0.41) remains bounded and positive, whereas the second term vanishes, and hence $f_{33}(\eta)$ is bounded and positive as $\eta \to 0^+$. Indeed, combining (SM0.41) and (SM0.42), we find that $2 \le \lim_{\eta\to 0^+} f_{33}(\eta) \le \frac{4}{\sqrt{4-\epsilon^2}}$. From this we deduce, by the arbitrariness of ϵ, that

$$\lim_{\eta\to 0^+} f_{33}(\eta) = 2. \tag{SM0.43}$$

To characterize the limit as $\eta \to \infty$, let $\epsilon > 0$ be fixed, and note that $1 \le 1 + 4\sin^2(\xi)/\eta^2 \le 1+\epsilon^2$ for all $\eta \ge 2/\epsilon$. In view of (SM0.40) we obtain

$$\frac{2\pi}{\eta[1+\epsilon^2]^{3/2}} \le \int_0^{\pi/2} \frac{4\eta^2}{[4\sin^2\xi+\eta^2]^{3/2}}\, d\xi \le \frac{2\pi}{\eta}. \tag{SM0.44}$$

The above shows that $f_{33}(\eta) \to 0$ as $\eta \to \infty$; specifically, by the arbitrariness of ϵ, we obtain

$$\lim_{\eta\to\infty} \frac{f_{33}(\eta)}{\frac{1}{\eta}} = 2\pi. \tag{SM0.45}$$

Using the same arguments as in (SM0.16)–(SM0.24), we obtain a pointwise limit result for the function $\mathcal{A}_{33}(x)$, namely

$$\lim_{r\to 0^+} \frac{\mathcal{A}_{33}(x)}{r\ln(\frac{1}{r})} = \begin{cases} \frac{1}{2}, & s_x \in (-L, L), \\ \frac{1}{4}, & s_x = \pm L. \end{cases} \tag{SM0.46}$$

All the above results derived under the convenient assumption $\theta_x = 0$ also hold for $\theta_x \neq 0$; the two cases differ by a simple rotation of coordinates about the axis of $\mathit{\Gamma}_{r,\mathrm{str}}$. Hence, by combining the above results with (SM0.4), and using the notation in (4.10), we obtain the following limit for each point $x = x_r(\theta_x, s_x) \in \mathit{\Gamma}_{r,\mathrm{str}}$,

$$\lim_{r\to 0^+} \frac{\mathcal{E}(x_r(\theta_x, s_x))}{r\ln(\frac{1}{r})} = \begin{cases} \frac{1}{2} g_{\mathrm{str}}^{-1}, & (\theta_x, s_x) \in [0, 2\pi) \times (-L, L), \\ \frac{1}{4} g_{\mathrm{str}}^{-1}, & (\theta_x, s_x) \in [0, 2\pi) \times \{\pm L\}. \end{cases} \tag{SM0.47}$$

To establish convergence in the L_1-norm, we again consider the integrals appearing in the scaled component functions $\mathcal{I}_{ij}(x)/[r\ln(\frac{1}{r})]$ and $\mathcal{A}_{ij}(x)/[r\ln(\frac{1}{r})]$. In view of the preceding developments, the integrals in these scaled functions are one of two basic forms, $H_+(x)$ or $H_-(x)$, where

$$H_+(x) = \frac{1}{\ln(\frac{1}{r})} \int_0^{\frac{1}{r}\sigma_{x,+}} h(\eta)\, d\eta, \qquad H_-(x) = \frac{1}{\ln(\frac{1}{r})} \int_0^{\frac{1}{r}|\sigma_{x,-}|} h(\eta)\, d\eta. \tag{SM0.48}$$

Here $h(\eta)$ is a continuous function for $\eta > 0$ that depends on the specific component $\mathcal{I}_{ij}(x)$ or $\mathcal{A}_{ij}(x)$, and has the general property

$$0 \le h(\eta) \le \begin{cases} C + C\ln(\frac{1}{\eta}), & 0 < \eta < 1, \\ C/\eta, & 1 \le \eta < \infty. \end{cases} \tag{SM0.49}$$

By combining (SM0.48) and (SM0.49) we find, by a direct evaluation of the integrals, and recalling the notation $x = x_r(\theta_x, s_x)$, $\sigma_{x,+} = L - s_x$ and $|\sigma_{x,-}| = L + s_x$,

$$\left.\begin{aligned} &0 \le H_+(x_r(\theta_x, s_x)) \le C + C|\ln(L - s_x)| \\ &0 \le H_-(x_r(\theta_x, s_x)) \le C + C|\ln(L + s_x)| \end{aligned}\right\} \quad (\theta_x, s_x) \in [0, 2\pi) \times (-L, L), \tag{SM0.50}$$

which holds for all $0 < r < \frac{1}{e}$ so that $\ln(\frac{1}{r}) > 1$. Since each of the functions $H_+(x_r(\theta_x, s_x))$ and $H_-(x_r(\theta_x, s_x))$ is dominated by a fixed integrable function, we note, by the Dominated Convergence Theorem, that $H_+(x_r(\theta_x, s_x))$ and $H_-(x_r(\theta_x, s_x))$ converge to their pointwise limits in the L_1-norm on $(\theta_x, s_x) \in [0, 2\pi) \times (-L, L)$ as $r \to 0^+$. From this we deduce that the limit in (SM0.47) converges in the L_1-norm.

Proof of Lemma 4.3. Given a curved cylindrical surface $\Gamma_{r,\mathrm{crv}}$, as defined in (4.13), let $\Gamma_{r,\mathrm{str}}$ be a corresponding straight cylindrical surface of the same radius and arclength, as defined in (4.9). We assume that the axial curve γ of $\Gamma_{r,\mathrm{crv}}$ is open so that $\gamma(-L) \neq \gamma(L)$; the case when γ is closed will be discussed later. Let $x_{\mathrm{crv}} = x_{r,\mathrm{crv}}(\theta_x, s_x)$ and $y_{\mathrm{crv}} = y_{r,\mathrm{crv}}(\theta_y, s_y)$ denote points on $\Gamma_{r,\mathrm{crv}}$, and let $x_{\mathrm{str}} = x_{r,\mathrm{str}}(\theta_x, s_x)$ and $y_{\mathrm{str}} = y_{r,\mathrm{str}}(\theta_y, s_y)$ denote points on $\Gamma_{r,\mathrm{str}}$. We will consider these surfaces for $r \in (0, \rho]$, where $\rho \in (0, a_\gamma)$ is a fixed constant, and $a_\gamma > 0$ denotes the injectivity radius of γ. For $r \in (0, \rho]$ we note that there is a one-to-one correspondence between $\Gamma_{r,\mathrm{str}}$ and $\Gamma_{r,\mathrm{crv}}$, which we indicate with the notation $x_{\mathrm{crv}} = x_{\mathrm{crv}}(x_{\mathrm{str}})$ and $y_{\mathrm{crv}} = y_{\mathrm{crv}}(y_{\mathrm{str}})$. The map from $\Gamma_{r,\mathrm{str}}$ to $\Gamma_{r,\mathrm{crv}}$ will be detailed below.

In what follows it will be useful to consider the ratio of chord lengths between pairs of points on $\Gamma_{r,\mathrm{str}}$ and the corresponding pairs on $\Gamma_{r,\mathrm{crv}}$. Specifically, for any $r \in (0, \rho]$ and $x_{\mathrm{str}}, y_{\mathrm{str}} \in \Gamma_{r,\mathrm{str}}$ with $x_{\mathrm{str}} \neq y_{\mathrm{str}}$, we consider the ratio

$$\omega(x_{\mathrm{str}}, y_{\mathrm{str}}) = \frac{|x_{\mathrm{str}} - y_{\mathrm{str}}|}{|x_{\mathrm{crv}}(x_{\mathrm{str}}) - y_{\mathrm{crv}}(y_{\mathrm{str}})|}. \tag{SM0.51}$$

From the Lipschitz continuity of γ, γ', and d_i, together with the fact that γ has a positive injectivity radius, we find that $\omega(x_{\mathrm{str}}, y_{\mathrm{str}})$ is uniformly bounded and positive; that is, there are positive constants C_1 and C_2 such that

$$C_1 \leq \omega(x_{\mathrm{str}}, y_{\mathrm{str}}) \leq C_2. \tag{SM0.52}$$

It will also be useful to have an explicit expression for the chord $(x_{\mathrm{crv}} - y_{\mathrm{crv}})$ in terms of the chord $(x_{\mathrm{str}} - y_{\mathrm{str}})$. Using the map from $\Gamma_{r,\mathrm{str}}$ to $\Gamma_{r,\mathrm{crv}}$, we have $x_{\mathrm{crv}} = x^1_{\mathrm{str}} d_1(x^3_{\mathrm{str}}) + x^2_{\mathrm{str}} d_2(x^3_{\mathrm{str}}) + \gamma(x^3_{\mathrm{str}})$ and $y_{\mathrm{crv}} = y^1_{\mathrm{str}} d_1(y^3_{\mathrm{str}}) + y^2_{\mathrm{str}} d_2(y^3_{\mathrm{str}}) + \gamma(y^3_{\mathrm{str}})$. From the Lipschitz continuity of γ, γ', and d_i, we note that each of the preceding quantities at y^3_{str} can be replaced with a quantity at x^3_{str} and a remainder term, which gives

$$(x_{\mathrm{crv}} - y_{\mathrm{crv}}) = \sum_{i=1}^{3} (x^i_{\mathrm{str}} - y^i_{\mathrm{str}}) d_i(s_x) - \sum_{\alpha=1}^{2} y^\alpha_{\mathrm{str}} R_\alpha(s_x, s_y)\sigma - R_0(s_x, s_y)\sigma^2, \tag{SM0.53}$$

where $y^3_{\mathrm{str}} = s_y$, $x^3_{\mathrm{str}} = s_x$, $\sigma = s_y - s_x$, and the coefficients $R_\alpha(s_x, s_y)$ and $R_0(s_x, s_y)$ of the remainder terms are defined and uniformly bounded for all $s_y \neq s_x$. Identifying the vectors $d_i(s_x)$ as the columns of an orthogonal matrix $Q(s_x) \in \mathbb{R}^{3\times 3}$, so that $d_i(s_x) = Q(s_x)e_i$, we note that $\sum_{i=1}^{3} (x^i_{\mathrm{str}} - y^i_{\mathrm{str}}) d_i(s_x) = Q(s_x)(x_{\mathrm{str}} - y_{\mathrm{str}})$. Taking the dot product of each side of (SM0.53) with itself, we obtain

$$\begin{aligned} &|x_{\mathrm{crv}} - y_{\mathrm{crv}}|^2 \\ &\quad = |x_{\mathrm{str}} - y_{\mathrm{str}}|^2 - 2\big(Q(s_x)(x_{\mathrm{str}} - y_{\mathrm{str}})\big) \cdot \big(y^\alpha_{\mathrm{str}} R_\alpha(s_x, s_y)\sigma + R_0(s_x, s_y)\sigma^2\big) \\ &\qquad + \big(y^\alpha_{\mathrm{str}} R_\alpha(s_x, s_y)\sigma + R_0(s_x, s_y)\sigma^2\big) \cdot \big(y^\beta_{\mathrm{str}} R_\beta(s_x, s_y)\sigma + R_0(s_x, s_y)\sigma^2\big). \end{aligned} \tag{SM0.54}$$

In the above, sums over $\alpha = 1, 2$ and $\beta = 1, 2$ are implied. Dividing through by $|x_{\mathrm{str}} - y_{\mathrm{str}}|^2$ and using (SM0.51), and noting that $y^1_{\mathrm{str}} = r\cos\theta_y$ and $y^2_{\mathrm{str}} = r\sin\theta_y$, and also noting that $|\sigma| \leq |x_{\mathrm{str}} - y_{\mathrm{str}}|$, we obtain the bound

$$\left|\frac{1}{\omega^2(x_{\mathrm{str}}, y_{\mathrm{str}})} - 1\right| \leq D_1 r + D_2 |x_{\mathrm{str}} - y_{\mathrm{str}}|, \tag{SM0.55}$$

where D_1 and D_2 are positive constants. Introducing the quantity $m = \omega - 1$, we find that $m = (\frac{\omega^2}{\omega+1})(1 - \frac{1}{\omega^2})$, and from (SM0.55) and (SM0.52) we obtain the important decomposition

$$\omega(x_{\rm str}, y_{\rm str}) = 1 + m(x_{\rm str}, y_{\rm str}), \tag{SM0.56}$$

where

$$|m(x_{\rm str}, y_{\rm str})| \le D_1 r + D_2 |x_{\rm str} - y_{\rm str}|. \tag{SM0.57}$$

All the above results hold for $r \in (0, \rho]$ and $x_{\rm str}, y_{\rm str} \in \mathit{\Gamma}_{r,{\rm str}}$ with $x_{\rm str} \neq y_{\rm str}$, where C_1, C_2, D_1 and D_2 are positive constants independent of $x_{\rm str}$, $y_{\rm str}$ and r.

In view of (4.2) and (4.17) we have

$$\mathcal{E}(x) = \mathcal{I}(x) + \mathcal{A}(x), \tag{SM0.58}$$

where

$$\mathcal{I}_{ij}(x) = \frac{1}{8\pi} \int_{\mathit{\Gamma}_{r,{\rm crv}}} \frac{\delta_{ij}}{|x-y|}\, dA_y, \quad \mathcal{A}_{ij}(x) = \frac{1}{8\pi} \int_{\mathit{\Gamma}_{r,{\rm crv}}} \frac{(x-y)_i (x-y)_j}{|x-y|^3}\, dA_y. \tag{SM0.59}$$

Similar to before, the components $\mathcal{I}_{ij}(x)$ and $\mathcal{A}_{ij}(x)$ are well-defined and continuous functions of $x \in \mathit{\Gamma}_{r,{\rm crv}}$ for each $r > 0$, and here we seek to characterize their limits as $r \to 0^+$.

We consider the first term in (SM0.58). To begin, let $x = x(x_{\rm str}) \in \mathit{\Gamma}_{r,{\rm crv}}$ be given, and at any point $y = y(y_{\rm str}) \in \mathit{\Gamma}_{r,{\rm crv}}$ note that $dA_y = J_r(y_{\rm str}) dA_{y_{\rm str}}$, where J_r is the Jacobian factor given in (4.14). Performing a change of variable in the integral, and using the chord ratio in (SM0.51), we get

$$\mathcal{I}_{ij}(x) = \frac{1}{8\pi} \int_{\mathit{\Gamma}_{r,{\rm str}}} \frac{\delta_{ij}\omega(x_{\rm str}, y_{\rm str})}{|x_{\rm str} - y_{\rm str}|}\, J_r(y_{\rm str}) dA_{y_{\rm str}}. \tag{SM0.60}$$

Moreover, introducing $\vartheta = \omega J_r - 1$, and using $\omega = 1 + m$, we obtain the relation $\vartheta = m + \omega(J_r - 1)$. In view of (SM0.52) and (SM0.57), and the fact that $|J_r - 1| \le Cr$ which follows from (4.14), we find

$$|\vartheta(x_{\rm str}, y_{\rm str})| \le D_1 r + D_2 |x_{\rm str} - y_{\rm str}|. \tag{SM0.61}$$

By definition of ϑ, the integral in (SM0.60) can be decomposed as

$$\mathcal{I}_{ij}(x) = \frac{1}{8\pi} \int_{\mathit{\Gamma}_{r,{\rm str}}} \frac{\delta_{ij}}{|x_{\rm str} - y_{\rm str}|}\, dA_{y_{\rm str}} + \frac{1}{8\pi} \int_{\mathit{\Gamma}_{r,{\rm str}}} \frac{\delta_{ij}\vartheta(x_{\rm str}, y_{\rm str})}{|x_{\rm str} - y_{\rm str}|}\, dA_{y_{\rm str}}. \tag{SM0.62}$$

Considering the scaled component functions $\mathcal{I}_{ij}(x)/[r\ln(\frac{1}{r})]$, and using the same type of arguments as in the proof of Lemma 4.2, we find that the second term vanishes in the limit $r \to 0^+$, whereas the first term is exactly the term considered in the straight case. Hence we obtain the following pointwise limit for each $x = x_{r,{\rm crv}}(\theta_x, s_x) \in \mathit{\Gamma}_{r,{\rm crv}}$, which as before converges in the L_1-norm, where $\mathrm{Id} \in \mathbb{R}^{3\times 3}$ denotes the identity,

$$\lim_{r\to 0^+} \frac{\mathcal{I}(x_{r,{\rm crv}}(\theta_x, s_x))}{r\ln(\frac{1}{r})} = \begin{cases} \frac{1}{2}\mathrm{Id}, & (\theta_x, s_x) \in [0, 2\pi) \times (-L, L), \\ \frac{1}{4}\mathrm{Id}, & (\theta_x, s_x) \in [0, 2\pi) \times \{\pm L\}. \end{cases} \tag{SM0.63}$$

We next consider the second term in (SM0.58). To begin, we use (SM0.53) to obtain the outer product relation

$$\begin{aligned}(x_{\mathrm{crv}} - y_{\mathrm{crv}}) \otimes (x_{\mathrm{crv}} - y_{\mathrm{crv}}) \\ = Q(s_x)\big[(x_{\mathrm{str}} - y_{\mathrm{str}}) \otimes (x_{\mathrm{str}} - y_{\mathrm{str}})\big]Q^T(s_x) + B(x_{\mathrm{str}}, y_{\mathrm{str}}),\end{aligned} \tag{SM0.64}$$

where

$$\begin{aligned}&B(x_{\mathrm{str}}, y_{\mathrm{str}}) \\ &= -2\,\mathrm{sym}\,\Big[\big(Q(s_x)(x_{\mathrm{str}} - y_{\mathrm{str}})\big) \otimes \big(y^\alpha_{\mathrm{str}} R_\alpha(s_x, s_y)\sigma + R_0(s_x, s_y)\sigma^2\big)\Big] \\ &\quad + \big(y^\alpha_{\mathrm{str}} R_\alpha(s_x, s_y)\sigma + R_0(s_x, s_y)\sigma^2\big) \otimes \big(y^\beta_{\mathrm{str}} R_\beta(s_x, s_y)\sigma + R_0(s_x, s_y)\sigma^2\big).\end{aligned} \tag{SM0.65}$$

In the above, sums over $\alpha = 1, 2$ and $\beta = 1, 2$ are implied, $Q(s_x)$ is an orthogonal matrix whose columns are $d_i(s_x)$, and $\mathrm{sym}\, A = \frac{1}{2}(A + A^T)$ for any square matrix A. Performing a change of variable in the second term of (SM0.58), and using (SM0.51), we get

$$\mathcal{A}(x) = \frac{1}{8\pi} \int_{\Gamma_{r,\mathrm{str}}} \frac{[(x_{\mathrm{crv}} - y_{\mathrm{crv}}) \otimes (x_{\mathrm{crv}} - y_{\mathrm{crv}})]\omega^3(x_{\mathrm{str}}, y_{\mathrm{str}})}{|x_{\mathrm{str}} - y_{\mathrm{str}}|^3}\, J_r(y_{\mathrm{str}}) dA_{y_{\mathrm{str}}}. \tag{SM0.66}$$

Using (SM0.64), and introducing $\zeta = \omega^3 J_r - 1$ similar to before, the above integral can be decomposed into three parts

$$\mathcal{A}(x) = \mathcal{A}^{(0)}(x) + \mathcal{A}^{(1)}(x) + \mathcal{A}^{(2)}(x), \tag{SM0.67}$$

where

$$\mathcal{A}^{(0)}(x) = \frac{1}{8\pi} \int_{\Gamma_{r,\mathrm{str}}} \frac{Q(s_x)[(x_{\mathrm{str}} - y_{\mathrm{str}}) \otimes (x_{\mathrm{str}} - y_{\mathrm{str}})]Q^T(s_x)}{|x_{\mathrm{str}} - y_{\mathrm{str}}|^3}\, dA_{y_{\mathrm{str}}}, \tag{SM0.68}$$

$$\mathcal{A}^{(1)}(x) = \frac{1}{8\pi} \int_{\Gamma_{r,\mathrm{str}}} \frac{B(x_{\mathrm{str}}, y_{\mathrm{str}})}{|x_{\mathrm{str}} - y_{\mathrm{str}}|^3}\, dA_{y_{\mathrm{str}}}, \tag{SM0.69}$$

$$\begin{aligned}\mathcal{A}^{(2)}(x) = \frac{1}{8\pi} \int_{\Gamma_{r,\mathrm{str}}} \Bigg[&\frac{Q(s_x)[(x_{\mathrm{str}} - y_{\mathrm{str}}) \otimes (x_{\mathrm{str}} - y_{\mathrm{str}})]Q^T(s_x)}{|x_{\mathrm{str}} - y_{\mathrm{str}}|^3} \\ &+ \frac{B(x_{\mathrm{str}}, y_{\mathrm{str}})}{|x_{\mathrm{str}} - y_{\mathrm{str}}|^3}\Bigg]\zeta(x_{\mathrm{str}}, y_{\mathrm{str}})\, dA_{y_{\mathrm{str}}}.\end{aligned} \tag{SM0.70}$$

We next consider the scaled function $\mathcal{A}(x)/[r \ln(\frac{1}{r})]$, decomposed as in (SM0.67), and examine the limit $r \to 0^+$. For the term $\mathcal{A}^{(0)}(x)$, we note that the integral is exactly as considered in the straight case in Lemma 4.2, up to a rotation $Q(s_x)$, which is fixed and can be taken outside of the integral and the limit. For the term $\mathcal{A}^{(1)}(x)$, we use the coordinate relations $y^1_{\mathrm{str}} = r\cos\theta_y$ and $y^2_{\mathrm{str}} = r\sin\theta_y$, and the inequality $|\sigma| \le |x_{\mathrm{str}} - y_{\mathrm{str}}|$, to note that the factor $B(x_{\mathrm{str}}, y_{\mathrm{str}})$ in (SM0.65) will contain terms of the following orders: $r|x_{\mathrm{str}} - y_{\mathrm{str}}|^2$, $|x_{\mathrm{str}} - y_{\mathrm{str}}|^3$, $r^2|x_{\mathrm{str}} - y_{\mathrm{str}}|^2$, $r|x_{\mathrm{str}} - y_{\mathrm{str}}|^3$, and $|x_{\mathrm{str}} - y_{\mathrm{str}}|^4$. For the terms of orders $r|x_{\mathrm{str}} - y_{\mathrm{str}}|^2$

and $r^2|x_{\mathrm{str}} - y_{\mathrm{str}}|^2$, we note that the factor of $|x_{\mathrm{str}} - y_{\mathrm{str}}|^2$ leads to weakly-singular integrals with a bounded limit as considered in detail in the proof of Lemma 4.2, and the additional factors of r and r^2 lead to vanishing limits. For the remaining terms of orders $|x_{\mathrm{str}} - y_{\mathrm{str}}|^3$, $r|x_{\mathrm{str}} - y_{\mathrm{str}}|^3$ and $|x_{\mathrm{str}} - y_{\mathrm{str}}|^4$, we note that the factor of $|x_{\mathrm{str}} - y_{\mathrm{str}}|^3$ is sufficient to lead to integrals that vanish in the limit. For the term $\mathcal{A}^{(2)}(x)$, we use the definition of ζ, along with (SM0.56) and (SM0.57), and the bounds $|J_r| \le C$ and $|J_r - 1| \le Cr$ which follow from (4.14), to obtain

$$|\zeta(x_{\mathrm{str}}, y_{\mathrm{str}})| \le D_1 r + D_2 |x_{\mathrm{str}} - y_{\mathrm{str}}|, \tag{SM0.71}$$

which implies that the integral vanishes in the limit. Hence, for the scaled function $\mathcal{A}(x)/[r\ln(\frac{1}{r})]$, we find that only the first term in (SM0.67) is non-vanishing in the limit $r \to 0^+$, and the result is a rotated version of the straight case considered in Lemma 4.2. Using the fact that $Q(s_x)[e_3 \otimes e_3]Q^T(s_x) = d_3(s_x) \otimes d_3(s_x)$, we obtain the following pointwise limit for each $x = x_{r,\mathrm{crv}}(\theta_x, s_x) \in \Gamma_{r,\mathrm{crv}}$, which as before converges in the L_1-norm,

$$\lim_{r\to 0^+} \frac{\mathcal{A}(x_{r,\mathrm{crv}}(\theta_x, s_x))}{r\ln(\frac{1}{r})} = \begin{cases} \frac{1}{2}(d_3 \otimes d_3)(s_x), & (\theta_x, s_x) \in [0, 2\pi) \times (-L, L), \\[2ex] \frac{1}{4}(d_3 \otimes d_3)(s_x), & (\theta_x, s_x) \in [0, 2\pi) \times \{\pm L\}. \end{cases} \tag{SM0.72}$$

By combining the results in (SM0.72) and (SM0.63) with (SM0.58), and using the relation $d_3(s_x) = \gamma'(s_x)$ and the notation in (4.16), we obtain the following limit for each point $x = x_{r,\mathrm{crv}}(\theta_x, s_x) \in \Gamma_{r,\mathrm{crv}}$, which converges in the L_1-norm,

$$\lim_{r\to 0^+} \frac{\mathcal{E}(x_{r,\mathrm{crv}}(\theta_x, s_x))}{r\ln(\frac{1}{r})} = \begin{cases} \frac{1}{2}g^{-1}(s_x), & (\theta_x, s_x) \in [0, 2\pi) \times (-L, L), \\[2ex] \frac{1}{4}g^{-1}(s_x), & (\theta_x, s_x) \in [0, 2\pi) \times \{\pm L\}. \end{cases} \tag{SM0.73}$$

In establishing the above result, we assumed that the curve γ was open with a Lipschitz unit tangent field, and also non-self-intersecting, so that its injectivity radius was positive. However, we note that the above result relies only on local properties of γ, and hence also applies to the case when the curve is closed, provided it has a Lipschitz unit tangent at the closure point, and is non-self-intersecting except for the closure point. In this case, the (one-sided) results for $s_x = -L$ and $s_x = L$ should be summed, and the value of the limit is $\frac{1}{2}g^{-1}(s_x)$ for all s_x.

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