Purpose: To learn what a graph and its adjacency matrix are, and to see how the powers of the adjacency matrix provide information about the graph and vice-versa.

Prerequisites: Matrix addition, multiplication.

Part I.

Definitions. A graph is a finite set of objects called nodes together with some paths between some of the nodes as illustrated in the figure below. A path of length one is a path that directly connects one node to another; for example, in the figure below, 1–2 is a path of length one from node 1 to node 2. A path of length \( k \) is a path made up of \( k \) consecutive paths of length one. The same length one path can appear more than once in a longer path; for example, in the figure below, 1–2–1 is a path of length two from node 1 to itself. Given a graph with \( m \) nodes, the adjacency matrix \( A \in M_{m,m} \) is defined by

\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots \\
a_{21} & a_{22} & a_{23} & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\]

where \( a_{ij} = \begin{cases} 1, & \exists \text{ path of length one between nodes } i, j \\ 0, & \text{otherwise} \end{cases} \).

For example, for the 6-node graph shown below, the adjacency matrix \( A \in M_{6,6} \) is

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
\end{pmatrix}.
\]

Theorem. Let \( A \) be the adjacency matrix of a graph and for any integer \( k \geq 1 \) let \( B = A^k \). Then \( b_{ij} \) is the number of paths of length \( k \) between nodes \( i, j \).

Problem 1. To understand why the theorem is true, consider the case \( k = 2 \) and let \( B = A^2 \) where \( A \) is the matrix above. In this case, the entries \( b_{ij} \) of \( B \) are

\[
b_{ij} = \sum_{r=1}^{6} a_{ir}a_{rj}, \text{ for example } b_{63} = a_{61}a_{13} + a_{62}a_{23} + a_{63}a_{33} + a_{64}a_{43} + a_{65}a_{53} + a_{66}a_{63}.
\]

What is the value of \( b_{66} \)? What does each term in the sum tell us about the number of paths of length two between nodes 6,3? (For example, \( a_{62}a_{23} = 1 \cdot 1 = 1 \) says that there are paths of length one between nodes 6,2 and 2,3, which gives one path of length two between 6,3.) Based on these observations, try to prove the above theorem by induction.

Problem 2. Use Matlab or a calculator to compute \( A^2 \) and \( A^3 \) for the the adjacency matrix \( A \) above. You should find that the (1,2)-element of \( A^2 \) is zero; this says that there are no paths of length two between nodes 1,2. Verify this by inspecting the graph. Similarly, you should find that the (6,6)-element of \( A^3 \) is two; this says that there are two paths of length three between nodes 6,6 (node 6 and itself). Using the graph verify that these paths are 6–4–5–6 and 6–5–4–6. In the same way, study the graph and the matrices \( A^2 \) and \( A^3 \) to answer the following questions: How many paths of length two are there between nodes 4,4? What are the paths? How many paths of length three are there between nodes 4,6? What are the paths?
Part II.

**Definition.** We will say that there is a *contact of level* $n$ between nodes $i, j$ on a graph if there is a path of length $k \leq n$ between nodes $i, j$.

**Problem 1.** Given any integer $n \geq 1$ let $C = A + A^2 + \cdots + A^n$. What is the significance of the elements $c_{ij}$ of $C$? In particular, what does it mean if $c_{ij} = 0$, and what does it mean if $c_{ij} > 0$? Is it possible to use $C$ to identify which nodes $i, j$ have a *contact of level* $n$ with each other?

**Problem 2.** Eight workers denoted $W_1, \ldots, W_8$ handle a dangerous substance. Safety precautions are taken, but accidents do happen. It is known that if a worker becomes contaminated with the substance, s/he could spread it through contact with another worker. The graph below shows which workers have direct contact with each other.

![Graph showing contact between workers](image)

(a) Find the adjacency matrix $A$ for the above graph. Use Matlab or a calculator to compute $C = A + A^2 + A^3$. Which workers have a *contact of level* 3 with $W_5$? Which workers have a *contact of level* 3 with $W_6$?

(b) Use Matlab or a calculator to find the smallest value of $n$ for which every worker has a *contact of level* $n$ with every other worker.

(c) Using your own mathematical definition of *dangerous* determine which workers are the most dangerous if contaminated. Which are the least dangerous? Explain your answers. (Whatever you say is okay as long as it is consistent with your definition of *dangerous*).