Homework 2

1) A linear $p$-step method for the initial value problem

\[ \frac{du}{dt}(t) = f(t, u(t)) \quad t > 0, \quad u(0) = u_0 \]

is a method of the form

\[ \sum_{\ell=0}^{p} a_{\ell} U_{k+\ell} = \Delta t \sum_{\ell=0}^{p} b_{\ell} f(t_{k+\ell}, U_{k+\ell}) \quad (1) \]

where $t_k = k\Delta t$ and $a_\ell, b_\ell$ are constants ($\ell = 0, \ldots, p$). Here we use polynomial interpolation to derive two families of methods of the form (1) and study some particular cases.

(a) For any $k \geq 0$ and $p \geq 1$ suppose $u(t)$ and $f(t, u(t))$ may be approximated on the interval $[t_k, t_{k+p}]$ by functions of the form

\[ v(t) = \sum_{\ell=0}^{p} q_{\ell}^{k,p}(t) U_{k+\ell} \quad \text{and} \quad g(t) = \sum_{\ell=0}^{p} q_{\ell}^{k,p}(t) f(t_{k+\ell}, U_{k+\ell}) \]

where $U_{k+\ell}$ is an approximation to $u(t_{k+\ell})$ and $q_{\ell}^{k,p}(t)$ are Lagrange polynomials defined by

\[ q_{\ell}^{k,p}(t) = \prod_{j=0}^{p} \frac{(t - t_{k+j})}{(t_{k+\ell} - t_{k+j})} \quad (\ell = 0, \ldots, p). \]

Show that $v(t_{k+s}) = U_{k+s}$ and $g(t_{k+s}) = f(t_{k+s}, U_{k+s})$ for any integer $0 \leq s \leq p$.

(b) Suppose we demand that the polynomials $v(t)$ and $g(t)$ satisfy the equation

\[ \frac{dv}{dt}(t_{k+p}) = g(t_{k+p}). \quad (2) \]

Show that (2) is equivalent to a linear multistep method of the form (1) for appropriate constants $a_\ell$ and $b_\ell$ ($\ell = 0, \ldots, p$). Methods of these type are called BDF (backward differentiation formulae) methods. Construct the two-step ($p = 2$) BDF method and show that it is convergent.

(c) Suppose we demand that the polynomials $v(t)$ and $g(t)$ satisfy the equation

\[ v(t_{k+p}) - v(t_{k+p-1}) = \int_{t_{k+p-1}}^{t_{k+p}} g(t) \, dt. \quad (3) \]

Show that (3) is equivalent to a linear multistep method of the form (1) for appropriate constants $a_\ell$ and $b_\ell$ ($\ell = 0, \ldots, p$). Methods of these type are called Adams-Moulton methods. Construct the two-step ($p = 2$) AM method and show that it is convergent.

(d) Considering the standard model problem with only real values $\lambda \leq 0$ show that the 2-step BDF method is unconditionally stable whereas the 2-step AM method is conditionally stable. In the latter case what restriction is imposed on the magnitude of $\lambda \Delta t$?

(e) Considering the standard model problem with arbitrary, complex $\lambda$ try to sketch a portion of the region of absolute stability of the 2-step BDF and AM methods. Hint: write $\lambda \Delta t$ in the form $x + iy$ and examine the modulus of each root $z$ (of stability polynomial) as a function of $x$ for various fixed $y$. (Matlab can perform complex arithmetic. For help type help i, help abs and help complex).
2) Consider the linear system of ordinary differential equations

\[ \begin{align*}
\dot{u}_1 &= -u_1, \\
\dot{u}_2 &= 4u_1 - 5u_2, \\
\end{align*} \]

with exact solution given by

\[ \begin{align*}
u_1(t) &= e^{-t}, \\
u_2(t) &= e^{-t} - e^{-5t}.
\end{align*} \]

Here we apply various linear two-step methods to (4) and investigate the roles of consistency and zero-stability in their convergence properties.

(a) Consider

\[ U_{k+2} = U_{k+1} + \Delta t \left[ \frac{3}{2} f_{k+1} - \frac{1}{2} f_k \right] \]

where \( f_k = f(t_k, U_k) \). Show that this method is consistent of order two and zero-stable and therefore convergent. Write a program to investigate the convergence properties of this method. In particular, consider the interval \([0, 2]\) and let \( E = \max_{0 \leq k \leq N} ||U_k - u(t_k)|| \) where \( u(t_k) \) is the exact solution at the grid point \( t_k = k\Delta t \) and \( \Delta t = 2/N \). Make a plot of \( \log(E) \) versus \( \log(\Delta t) \) and a plot of \( E \) versus \( \Delta t \) as before. Does \( E \) tend to zero at the predicted rate as \( \Delta t \) tends to zero?

(b) Consider

\[ U_{k+2} = U_{k+1} + \Delta t \left[ \frac{3}{2} f_{k+1} - \frac{1}{3} f_k \right]. \]

Show that this method is zero-stable but not consistent and therefore not convergent. (A method is inconsistent if the local truncation error \( \tau \) does not tend to zero as \( \Delta t \) tends to zero.) Make a plot of \( \log(E) \) versus \( \log(\Delta t) \) and a plot of \( E \) versus \( \Delta t \) as before. Does \( E \) tend to zero as \( \Delta t \) tends to zero? Is the behavior consistent with the basic error bound implied by zero-stability?

(c) Consider

\[ U_{k+2} = \frac{17}{16} U_k - \frac{1}{16} U_{k+1} + \Delta t \left[ \frac{33}{32} f_{k+1} + \frac{33}{32} f_k \right]. \]

Show that this method is consistent of order one but not zero-stable and therefore not convergent. To see what is happening with this method you need only make a plot of \( \log(E) \) versus \( \log(\Delta t) \) for moderate values of \( N \) (up to \( N = 400 \) or so). What is happening to \( E \) as \( \Delta t \) tends to zero? Is the behavior consistent with the meaning of zero-stability?