Proof or non-Proof?

Note: I've fixed all the typos that I remember noticing in class! I've probably introduced new ones in the solutions, though...

The following are four attempted proofs of the statement: "If $AB = BA$, then A and B are square." Some of them are fully correct, and some are not (I'm not saying how many!) Which ones are which? What's wrong with the ones that aren't proofs? Feel free to work in groups!

Proof 1:
We will do this proof by contrapositive. Accordingly, here's what we're proving:

Assumptions: $\boxed{A \text{ and } B \text{ are both not square.}}$
Need to show: $AB \neq BA$.

Let A be $m \times \boxed{n}$, and B be $\boxed{n} \times p$. Then,

$$AB \text{ is } m \times p$$
$$BA \text{ is } n \times n$$

Since A is not square, $m \neq n$. Therefore, the number of rows of AB is not equal to the number of rows of BA, and hence $AB \neq BA$, as required. □

Issues:

1. That is not the correct statement of the contrapositive – the correct negation of "A and B are square" is "One of A and B is not square."

2. Why are we assuming that the number of columns of A is equal to the number of rows of B?

Proof 2:
Assumptions: $AB = BA$
Need to show: A and B are both square.

Let A be $m \times \boxed{n}$, and B be $\boxed{n} \times p$. Then, since BA is defined, we must have $p = m$. Therefore, A is $m \times n$ and B is $n \times m$. Therefore,

$$AB \text{ is } m \times m$$
$$BA \text{ is } n \times n$$

Since those are equal, we must have $m = n$. Thus, A and B are both $n \times n$ and hence are square, as required. □

Issues:

1. Again, why are we assuming that the number of columns of A is equal to the number of rows of B? In a basic proof like this, we need to justify that!

Proof 3:
Assumptions: $AB = BA$
Need to show: A and B are both square.

Let A be $m \times n$, and B be $p \times q$. Since AB is defined, $n = p$. Since BA is defined, $q = m$. Therefore, we have that B is $n \times m$. Thus,

$$AB \text{ is } m \times m$$
$$BA \text{ is } n \times n$$

Since those are equal, we must have $m = n$. Thus, A and B are both $n \times n$ and hence are square, as required. $\square$

This is a correct proof!

Proof 4:
Assumptions: $AB = BA$
Need to show: A and B are both square.

Let A be $m \times n$, and B be $p \times q$. Since AB is defined, $n = p$. Since BA is defined, $q = n$. Therefore, we have that B is $n \times m$. Thus,

$$AB \text{ is } m \times m$$
$$BA \text{ is } n \times n$$

$\boxed{\text{Therefore, } AB \text{ and } BA \text{ are both square, so we're done.}}$ $\square$

Issues:

1. This is a perfectly logical sequence of steps, but it's concluding statement isn't right! We weren't supposed to show that AB and BA are square – we were supposed to show that A and B are square.

The Joy of Sets

Vectors:

For each of the following sets, do the following:

- Give an example of an element of the set (or show that it's empty.)
- Check whether the column vector $\vec{x} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ is in it.

Hint: at least one of the above two tasks should be easy for each set!

1.
$$S_1 = \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} \text{ orthogonal to } [1,0,0]^T\}$$

   - A vector $[x_1, x_2, x_3]^T$ in in the set if
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$$
which works out precisely to $x_1 = 0$. Thus, an example of an element in this set is $\boxed{[0,1,1]^T}$.
   - Checking, $[1,1,1]^T \cdot [0,0,1]^T = 1$, and therefore $[1,1,1]^T$ is not orthogonal to $[1,0,0]^T$. Thus, $[1,1,1]^T$ is $\boxed{\text{not}}$ in the set.

2.
$$S_2 = \{\vec{x} \in \mathbb{R}^3 \mid A\vec{x} = \vec{0}\}$$
where $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & -3 \end{bmatrix}$.

   - A vector $[x_1, x_2, x_3]^T$ in in the set if
$$A\vec{x} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
which works out to
$$x_1 - x_3 = 0$$
$$x_1 + 2x_2 - 3x_3 = 0$$
Rewriting this as an augmented matrix and row-reducing, this works out to $x_1 = x_3, x_2 = x_3$, and therefore all solutions are of the form $[x_3, x_3, x_3]$. Thus, an example of an element in this set is $\boxed{[-1,-1,-1]^T}$.
   - Checking, $A[1,1,1]^T = [0,0]^T$, therefore $[1,1,1]^T$ $\boxed{\text{is}}$ in the set. (You can also notice that it matches the general form found previously.)

3.
$$S_3 = \{c_1\vec{v} + c_2\vec{w} \mid c_1, c_2 \in \mathbb{R}\}$$
where $\vec{v} = [2,4,6]^T, \vec{w} = [0,1,2]^T$.

- A vector $[x_1, x_2, x_3]^T$ in in the set if
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1\vec{v} + c_2\vec{w} = c_1 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$
for some choice of scalars c_1 and c_2. Thus, picking arbitrary c_1 and c_2 provides us with an example of a member of the set. Let's pick $c_1 = 1/2, c_2 = 0$. Then, $\vec{x} = \boxed{[1,2,3]^T}$ is an example of an element in the set.
- Here, the questions is whether $[1,1,1]^T$ can be written as $c_1\vec{v} + c_2\vec{w}$ for some choice of c_1, c_2. Setting it up, we need to see if there exist solutions c_1, c_2 to the following system:
$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2c_1 \\ 4c_1 + c_2 \\ 6c_1 + 2c_2 \end{bmatrix}$$
This is simply a system of linear equations. Solving, we get that $c_1 = 1/2, c_2 = -1$, so
$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1/2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$
and therefore $[1,1,1]^T$ $\boxed{\text{is}}$ in the set.

4.
$$S_4 = \{\vec{x} \in \mathbb{R}^4 \mid \|\vec{x}\| = \sqrt{3}\}$$

- A vector $\vec{x} = [x_1, x_2, x_3, x_4]^T$ in in the set if
$$\sqrt{3} = \|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$$
which simplifies precisely to $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 3$. Therefore, an example of a member of this set is $\boxed{[1,0,1,1]^T}$.
- Since $[1,1,1]^T$ is not in $\mathbb{R}^4$, it is $\boxed{\text{not}}$ a member of this set.

5.
$$S_5 = \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} \text{ in the column space of } A\}$$
where $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \\ 6 & 2 \end{bmatrix}$.

- A vector is in the column space of a matrix if it can be written as a linear combination of the columns. Therefore, $\vec{x} = [x_1, x_2, x_3]^T$ is in this set if
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$
for some scalars c_1, c_2. It should be easy to see that this is the same condition as for the set S_3 above, and therefore $S_5 = S_3$ and all our reasoning for that set applies here. Therefore, $\boxed{[1,2,3]^T}$ is an example of an element of this set, and
- $[1,1,1]^T$ $\boxed{\text{is}}$ an element of the set.

Matrices:

For each of the following sets, do the following:

- Give an example of an element of the set (or show that it's empty.)
- Check whether the matrix
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$
is in it.

Hint: as above, one of the above two tasks should be easy for each set!

1.
$$R_1 = \{C \mid \text{rank}(C) = 2\}$$
   - The rank of a matrix is the number of non-zero rows in its row-reduced echelon form. Accordingly, an example of a matrix in R_1 is
$$\boxed{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}}$$
(For simplicity, I chose a matrix which is already row-reduced.)
   - The rref of A is
$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$
Clearly, $\text{rank}(A) = 2$, so A $\boxed{\text{is}}$ in R_1.

2.
$$R_2 = \{C \mid C \text{ is a } 2 \times 3 \text{ matrix, } \text{rank}(C) = 3\}$$
   - Since the rank of a matrix is the number of nonzero rows in its rref, and a 2×3 matrix only has 2 rows, it's impossible for its rank to be 3. Hence, this set is $\boxed{\text{empty.}}$

- Since the set is empty, A is not in it.

3.
$$R_3 = \{C \mid C \text{ is row equivalent to } B\}$$

where $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$.

- Any matrix produced from B by row operations is row equivalent to B. Therefore, doing the row operation $R_2 : R_1 + R_2$ we get the following matrix which is in the set:

$$\boxed{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix}}$$

- As we learned in class, two matrices are row-equivalent if and only if that have the same rref. Since the rref if A is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

and the rref of B is B (it's already row-reduced), it follows that A and B are not row-equivalent. Therefore, A is $\boxed{\text{not}}$ in the set.

4.
$$R_4 = \left\{ C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a + b = 0, c + d = 0 \right\}$$

- We just need to pick a, b, c, d to satisfy the constraints. Letting $a = 1, c = 0$ we get the following matrix as an example:

$$\boxed{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}}$$

- A isn't even 2×2: it's clearly $\boxed{\text{not}}$ in the set.

5.
$$R_5 = \left\{C \mid C \text{ symmetric, } C^{-1} \text{ has first row equal to } [1, 1, 1]\right\}$$

- As seen in the homework, C is symmetric if and only if C^{-1} is symmetric. Furthermore, given C^{-1} we can calculate C by taking the inverse of C^{-1}. Therefore, we can take a symmetric C^{-1} whose first row is $[1, 1, 1]$ and invert it to get an example of a matrix in the set. For example, we can take

$$C^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Inverting, we get

$$C = \boxed{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}}$$

which is clearly an example of a matrix in the set. (It's clearly symmetric, and we chose its inverse to have $[1, 1, 1]$ as the first row.)

- A is not square and hence has no inverse. Therefore, it is $\boxed{\text{not}}$ in the set.

Contrapositive Proofs

Do the following proofs by contrapositive:

1. Prove that if $\vec{x} \cdot \vec{y} \neq 0$, then $\|\vec{x}\|^2 + \|\vec{y}\|^2 \neq \|\vec{x} + \vec{y}\|^2$.

   Proof:
   Since we're doing the contrapositive, instead of proving A implies B, we prove that (not B) implies (not A). Here, the statements A and B are as follows:

   $$\begin{aligned} &A : \vec{x} \cdot \vec{y} \neq 0 \\ &B : \|\vec{x}\|^2 + \|\vec{y}\|^2 \neq \|\vec{x} + \vec{y}\|^2 \end{aligned}$$

   These are easy to negate, resulting in the following set up:

   Assumptions: $\|\vec{x}\|^2 + \|\vec{y}\|^2 = \|\vec{x} + \vec{y}\|^2$.
   Need to show: $\vec{x} \cdot \vec{y} = 0$

   We are given that
   $$\|\vec{x}\|^2 + \|\vec{y}\|^2 = \|\vec{x} + \vec{y}\|^2$$
   Rewriting everything in terms of dot products and simplifying:

   $$\begin{aligned} \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y} &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} \\ &\Rightarrow 0 = 2\vec{x} \cdot \vec{y} \\ &\Rightarrow 0 = \vec{x} \cdot \vec{y} \end{aligned}$$

   as required. $\square$

2. Prove that if A is a non-zero 2×2 matrix, then either $A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is non-zero.

   Proof:
   We need to negate both statements, like in the first question. Note that

   $$\text{not}\left(\text{either } A \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ or } A \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ is non-zero}\right) = \left(A \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } A \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ are both } \vec{0}\right)$$

   Therefore, the set up is as follows:
   Assumptions: A is 2×2, $A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are both $\vec{0}$.
   Need to show: A is the zero matrix.

   Let
   $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then the assumption says that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

and that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

These clearly combine to state that $a = 0, c = 0, b = 0$ and $d = 0$, hence A is the zero matrix, as required. □

3. Prove that if $\vec{x} \in \mathbb{R}^n$, and $A\vec{x} = \vec{0}$ for every $n \times n$ matrix A, then $\vec{x}$ is the zero vector.

   Proof:
   Assumptions: $\vec{x} \in \mathbb{R}^n$, and $\vec{x} \neq \vec{0}$.
   Need to show: There exists an $n \times n$ matrix A such that $A\vec{x} \neq \vec{0}$.

   Let A be the $n \times n$ matrix all of whose rows are equal to $\vec{x}^T$. Then, by the rules of matrix multiplication,

   $$\text{Entry } i \text{ of } A\vec{x} = (\text{Row } i \text{ of } A) \cdot \vec{x} = \vec{x} \cdot \vec{x} = \|\vec{x}\|^2$$

   Therefore,

   $$A\vec{x} = \begin{bmatrix} \|\vec{x}\|^2 \\ \|\vec{x}\|^2 \\ \vdots \\ \|\vec{x}\|^2 \end{bmatrix} \neq \vec{0}$$

   so we're done.

   □

4. Prove that if A is a singular $n \times n$ matrix, then $A^2 \neq I_n$.

   Proof:
   Assumptions: A is $n \times n$, and $A^2 = I_n$.
   Need to show: A is non-singular.

   As we know, to show that B is the inverse of A (provided both are $n \times n$), we just need to check that $AB = I_n$. Now, since we have that $A \cdot A = A^2 = I_n$, we can conclude that A is the inverse of A, and hence A is non-singular. □

5. **Challenge:** Assume that A is an $m \times n$ matrix and B is an $n \times m$ matrix such that $AB = I_m$ and $BA = I_n$. Show that $m = n$.

   Hint: Due to symmetry, $m < n$ and $n < m$ are equivalent. So we can assume that $m < n$. This can be used to show that there exists a non-trivial solution to $A\vec{x} = \vec{0}$. Show that this implies that $BA \neq I_n$.

   Proof:
   Assumptions: $m \neq n$.
   Need to show: Either $AB \neq I_m$ or $BA \neq I_n$.

   There are two possibilities for $m \neq n$: either $m < n$ or $n < m$. Let us first consider the case $m < n$, and show that $BA \neq I_n$.

   Since $m < n$, the system of equations $A\vec{x} = \vec{0}$ has more variables than equations, and therefore there exists a a non-trivial solution $\vec{x}_0$. Thus, $A\vec{x}_0 = \vec{0}$, so
   $$(BA)\vec{x}_0 = B(A\vec{x}_0) = A\vec{0} = \vec{0}$$

   Hence, $\vec{x}_0$ is a nontrivial solution to $(BA)\vec{x} = \vec{0}$. This means that $BA \neq I_n$, as $I_n\vec{x} = \vec{0}$ only has the trivial solution.

   Now, consider the case $n < m$. It should be clear that the above argument with the roles of A and B switched will result in showing that $AB \neq I_m$. (This is what is meant by the cases $m < n$ and $n < m$ are entirely symmetric.) Thus, in both cases we can show that either $AB \neq I_m$ or $BA \neq I_n$, so we're done. $\square$