Final Exam: Concepts to Review

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The final covers everything we did in class so far – it is cumulative. However, there will be more emphasis on the things covered since the last midterm: that is, half of Section 3.4, and Sections 4.1, 4.2, 4.3, 4.4, 4.5, 5.1, and a bit of Section 5.2.

This review sheet incorporates the earlier review sheets for the midterms.

1. Fundamental Operations with Vectors (Section 1.1)
   - Addition of vectors, multiplying vectors by a scalar.
   - The length $\|\vec{x}\|$ of a vector:
     $$\|[x_1, x_2, \ldots, x_n]\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$
   - Unit vectors; finding the unit vector in the same direction as a vector $\vec{x}$.
   - Using the properties of addition and scalar multiplication (Theorem 1.3)
   - Definition of a linear combination: a vector $\vec{v}$ is a linear combination of vectors $\vec{v}_1, \ldots, \vec{v}_n$ if it's possible to find scalars $c_1, c_2, \ldots, c_n$ such that
     $$\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$
2. The dot product (Section 1.5)
   - The definition of the dot product of $\vec{x} = [x_1, x_2, \ldots, x_n]$ and $\vec{y} = [y_1, y_2, \ldots, y_n]$:
     $$\begin{aligned}\vec{x} \cdot \vec{y} &= [x_1, x_2, \ldots, x_n] \cdot [y_1, y_2, \ldots, y_n] \\ &= x_1y_1 + x_2y_2 + \cdots + x_ny_n\end{aligned}$$
   - The Cauchy-Schwarz inequality: $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$.
   - The triangle inequality: $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$.
   - The angle between two vectors: if θ is the angle between $\vec{x}$ and $\vec{y}$, then θ is defined to be between 0 and π and it satisfies
     $$\vec{x} \cdot \vec{y} = \|\vec{x}\|\,\|\vec{y}\|\cos(\theta)$$

- Using the dot-product to check for orthogonality, being parallel, being in opposite directions
- The definition of a projection of $\vec{b}$ onto $\vec{a}$, denoted by $\text{proj}_{\vec{a}}\vec{b}$:

$$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \right) \vec{a}$$

3. Fundamental Operations with Matrices (Section 1.4)

- Definition of an $m \times n$ matrix
- Matrix addition, multiplying matrices by scalars
- Definitions of special matrices: square matrices, diagonal matrices, identity matrices, upper triangular matrices, lower triangular matrices, zero matrices
- Properties of addition and scalar multiplication of matrices (Theorem 1.11)
- The transpose of a matrix and its properties

4. Matrix Multiplication (Section 1.5)

- How to multiply matrices: if A is an $m \times n$ matrix and B is an $n \times p$ matrix, then AB is an $m \times p$ matrix such that

$$(i, j) \text{ entry of } AB = (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B)$$

  where the $\cdot$ means the dot product
- Matrix multiplication is **not** commutative: we rarely have $AB = BA$.
- The following identities:

$$k\text{th column of } AB = A(k\text{th column of } B)$$
$$k\text{th row of } AB = (k\text{th row of } A)B$$

- Fundamental properties of matrix multiplication (Theorem 1.14)
- Powers of square matrices
- Matrix multiplication and transposes: $(AB)^T = B^T A^T$.
- Linear combinations from matrix multiplication: how to find a linear combination of the rows of A by multiplying A on the left by a row vector, how to find a linear combination of the columns of A by multiplying A on the right by a column vector

5. Systems of Equations (Sections 2.1 and 2.2.)

- Writing down a system of linear equations in augmented matrix form
- Getting the system into row-reduced echelon form

- Using the row-reduced echelon form to find all solutions to the system
- The criteria for when a system in row-reduced form has
  - No solutions
  - Exactly one solution
  - Infinitely many solutions
- What the system $A\vec{x} = \vec{b}$ corresponds to as an augmented system of equations: that is, $[A|\vec{b}]$.
- If R is a row operation, then $R(AB) = R(A)B$.
- What a homogoneous system is: that is, the system $A\vec{x} = \vec{0}$.

6. Equivalent Systems, Rank, and Row Space (Section 2.3)

   - Two systems are *equivalent* if they have exactly the same solutions.
   - A matrix C is *row equivalent* to a matrix D if C is obtained from C with a finite number of row operations.
   - Knowing how to 'reverse' row operations: this can be used to show that if C is row equivalent to D, then D is row equivalent to C.
   - How to test if two matrices are row equivalent: two matrices are row equivalent if they have the exact same reduced row echelon form
   - The definition of the *rank* of A:

$$\begin{aligned}\text{rank}(A) &= \{\text{number of non-zero rows in the rref of } A\} \\ &= \{\text{number of pivotal columns in the rref of } A\}\end{aligned}$$

   - When a homogeneous system (that is, the system $A\vec{x} = \vec{0}$) has one or infinitely many solutions:
     (a) If $\text{rank}(A) < n$, then the system has a non-trivial solution
     (b) If $\text{rank}(A) = n$, then the system has the one solution $\vec{x} = \vec{0}$.
   - The definition of the *row space* of a matrix, and how to check whether a vector is in the row space: the row space of A is the set of linear combinations of the rows of A.
   - If A is row equivalent to B, then the row space of A is equal to the row space of B.

7. Inverses of Matrices (Section 2.4)

   - For a square $n \times n$ matrix A, B is the *inverse* of A if

$$AB = BA = I_n$$

     This is denoted by $B = A^{-1}$.
   - If A and B are $n \times n$ matrices such that $AB = I_n$, then we have that A and B are inverses: that is, $BA = I_n$.

- A square matrix is *singular* if and only if it does not have an inverse. If it does have an inverse, then it is *nonsingular*.
- Inverse properties: Let A and B be nonsingular $n \times n$ matrices. Then,
  (a) A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.
  (b) AB is nonsingular, and $(AB)^{-1} = B^{-1}A^{-1}$
  (c) A^T is nonsingular and $(A^T)^{-1} = (A^{-1})^T$.
- For a 2×2 matrix A,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  (In fact, you now know that the $\frac{1}{ad-bc}$ term corresponds to $\frac{1}{|A|}$!)
- How to find the inverse of any matrix:
  (a) If A is $n \times n$, then augment A with I_n, getting the system $[A|I_n]$.
  (b) Row reduce A, performing the same operations on both sides of the augmenting bar
  (c) If A row reduces to I_n, then it has an inverse, and the end result is $[I_n|A^{-1}]$.
  (d) If the row reduced echelon form of A is not I_n, then A has no inverse: it is singular.
- An $n \times n$ matrix A is nonsingular if and only if $\text{rank}(A) = n$.
- The system $A\vec{x} = \vec{b}$ has a unique solution if and only if A is nonsingular. If A is singular, then it either has no solutions or infinitely many solutions.

8. Determinants (Sections 3.1, 3.2, and 3.3 – we did it out of order!)

- For a 2×2 matrix, the determinant is calculated as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- For bigger matrices, the determinant is calculated using row or column expansion. Before defining this, define A_{ij} to be the matrix we get by crossing out row i and column j of A. Then,

$$\mathcal{A}_{ij} = (-1)^{i+j}|A_{ij}|$$

  This is called the (i, j) cofactor of A, while $|A_{ij}|$ is the (i, j) minor of A.
- Calculating the determinant using row expansion: if A is an $n \times n$ matrix, then for any i,

$$\begin{aligned} |A| &= a_{i1}\mathcal{A}_{i1} + a_{i2}\mathcal{A}_{i2} + \cdots + a_{i1}\mathcal{A}_{i1} \\ &= (-1)^{i+1}a_{i1}\,|A_{i1}| + (-1)^{i+2}a_{i2}\,|A_{i1}| + \cdots + (-1)^{i+n}a_{in}\,|A_{in}| \end{aligned}$$

- Calculating the determinant using column expansion: if A is an $n \times n$ matrix, then for any j,

$$\begin{aligned} |A| &= a_{1j}\mathcal{A}_{1j} + a_{2j}\mathcal{A}_{i2} + \cdots + a_{nj}\mathcal{A}_{nj} \\ &= (-1)^{j+1}a_{1j}\,|A_{1j}| + (-1)^{j+2}a_{2j}\,|A_{1j}| + \cdots + (-1)^{j+n}a_{nj}\,|A_{nj}| \end{aligned}$$

9. Properties of Determinants (Sections 3.1, 3.2, and 3.3)

   - $|AB| = |A||B|$ for all $n \times n$ matrix A and B.
   - $|A^T| = |A|$.
   - However, it's not true that $|A + B| = |A| + |B|$ in general!
   - $|A| = 0$ if and only if A is singular – that is, doesn't have an inverse.
   - Effects of row operations on the determinant (summary posted as solutions of the in-class work.)
   - For an upper triangular $n \times n$ matrix A,

$$|A| = a_{11}a_{22}\cdots a_{nn}$$

   - Using row operations to calculate determinants
   - If A is an $n \times n$ matrix, then the following statements are all equivalent (all imply each other):
     (a) A is singular (doesn't have an inverse)
     (b) rank$(A) < n$
     (c) $|A| = 0$
     (d) $A\vec{x} = \vec{0}$ has a nontrivial solution.
     (e) $A\vec{x} = \vec{b}$ either has no solutions or infinitely many solutions (depending on $\vec{b}$.)
   - Similarly, if A is an $n \times n$ matrix, then the following statements are all equivalent (all imply each other):
     (a) A is nonsingular (has an inverse)
     (b) rank$(A) = n$
     (c) $|A| \neq 0$
     (d) $A\vec{x} = \vec{0}$ only has the trivial solution $\vec{x} = \vec{0}$.
     (e) $A\vec{x} = \vec{b}$ has exactly one solution for all $\vec{b}$.

10. Sets: using set notation, giving example of elements of sets and checking whether things are in sets.

11. Eigenvalues and Eigenvectors (Section 3.4)

   - λ is an eigenvalue of A if there exists a nonzero $\vec{x}$ such that $A\vec{x} = \lambda x$. In that case, $\vec{x}$ is an eigenvector of A with eigenvalue λ.

- The characteristic polynomial of A is defined to be

$$p_A(x) = |xI_n - A|$$

- To find eigenvalues, solve $0 = p_A(x)$.
- The eigenspace E_λ is $\{\vec{x} \mid A\vec{x} = \lambda\vec{x}\}$. To find all vectors in E_λ, solve

$$(\lambda I_n - A)\vec{x} = \vec{0}$$

- A matrix B is similar to a matrix A if there exists some (nonsingular) matrix P such that $P^{-1}AP = B$.
- Diagonalizing a matrix means finding a diagonal matrix D such that A is similar to D. Here's the algorithm for an $n \times n$ matrix A:
  (a) Find the eigenvalues of A.
  (b) For each eigenvalue λ of A, write E_λ as the span of a number of vectors. These vectors will be called the fundamental eigenvectors for λ. (Indeed, to be a set of fundamental eigenvectors, the vectors have to be linearly independent. But doing it the way we've learned in class always results in that!)
  (c) Count the total number of fundamental eigenvectors you got from all the eigenvalues. If you got fewer than n, the matrix is not diagonalizable.
  (d) If you get precisely n fundamental eigenvectors, then let P be the matrix whose ith column is the ith fundamental eigenvector. Then, we have that $P^{-1}AP = D$, where D is a diagonal matrix with the ith entry on the diagonal corresponding to the eigenvalue of the ith column of P.
- The algebraic multiplicity of an eigenvalue λ of A is the highest power of $(x - \lambda)$ that divides the characteristic polynomial $p_A(x)$. We have the following inequality:

$$1 \leq \# \text{ of fundamental e.vectors of } \lambda \leq \text{alg. multiplicity of } \lambda$$

- A matrix is diagonalizable precisely if the number of fundamental eigenvectors corresponding to λ is equal to the algebraic multiplicity of λ for all λ.
- Using diagonalization to compute large powers of a matrix (you saw this on the homework.)

12. Vector Spaces (Section 4.1)

- Checking the ten properties of vector spaces to see whether something is or is not a vector space. (The properties are on page 204 – it'll take me too long to copy them!)
- Using the properties to prove identities such as $0\vec{v} = \vec{v}$.

- Definitions of some common vector spaces: $\mathbb{R}^n, \mathcal{M}_{mn}$, vector spaces of functions

13. Subspaces (Section 4.2)

    - W is a subspace of a vector space V if it's a non-empty subset of V (that is, every element of W is contained in V, and W contains at least one element), and if it's closed under vector addition and scalar multiplication.
    - To be more precise, we need to have that for all $\vec{x}, \vec{y} \in W$, and for all scalars c,
      (1) $\vec{x} + \vec{y}$ is in W
      (2) $c\vec{x}$ is in W
    - Knowing how to check whether something is a subspace

14. Span (Section 4.3)

    - Let S be a subset of a vector space V. Then $\vec{v}$ is a finite linear combination of the vectors in S if we can write

      $$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$$

      for some vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in S and some scalars $a_1, a_2, \ldots, a_n$.
    - The span of S is the set of all finite linear combinations of S. This is written as $\text{span}(S)$.
    - $\text{span}(\{\})$ is defined to be $\{\vec{0}\}$.
    - Method for simplifying span: Let S be a finite subset of $\mathbb{R}^n$ containing k vectors. Then, a way to write $\text{span}(S)$ in a simpler way is as follows:
      (1) Make a matrix A whose rows are the elements of S. In that case, the span of S is just the row space of A.
      (2) Find the row-reduced echelon form of A, call it B. Since row operations do not change the row space, $\text{span}(S)$ is the row space of B. Since the zero rows of B don't contribute anything to the row space, we see that the simplified form of $\text{span}(S)$ is

      $$\text{span}(S) = \text{span}(\text{non-zero rows of } B)$$

15. Linear Dependendence and Independence (Section 4.4):

    - Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a finite subset of a vector space V. Then, S is linearly dependent if there exist scalars $a_1, a_2, \ldots, a_n$ which aren't all 0, such that

      $$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$$

    - S is linearly independent if it is NOT linearly dependent.

- Rephrasing, S is linearly independent if $a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$ implies that $a_1 = a_2 = \cdots = a_n = 0$.
- Test for linear independence: let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a finite subset of $\mathbb{R}^n$.
  (1) Let A be a matrix whose columns are the elements of S. Then, the equation $a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$ is in augmented matrix form $[A \,|\, \vec{0}\,]$.
  (2) Find B, the row-reduced echelon form of A. If B has a pivot in every column, then the system $[B \,|\, \vec{0}\,]$ which is equivalent to $[A \,|\, \vec{0}\,]$ only has the solution $a_1 = a_2 = \cdots = a_n = 0$, so S is linearly independent. Otherwise, S is linearly dependent.
- If $k > n$, then any set of k vectors in $\mathbb{R}^n$ is linearly dependent. This is easy to see because in the set up above, the system $[A \,|\, \vec{0}\,]$ will have more columns than rows, and hence will have a column without a pivot.
- An infinite set S is linearly dependent if there exists a finite subset of S which is linearly dependent. An infinite set S is linearly independent if every finite subset of S is linearly independent.

16. Basis and Dimension (Section 4.5)

- Let $\mathcal{B}$ be a subset of a vector space V. Then $\mathcal{B}$ is a basis for V if the span of $\mathcal{B}$ is V (often stated as $\mathcal{B}$ spans V), and if $\mathcal{B}$ is linearly independent.
- If V has a basis $\mathcal{B}$ with finitely many elements, then all bases of V have exactly the same number of elements: that is, if $\mathcal{B}_1$ is a different basis of V, then $|\mathcal{B}_1| = |\mathcal{B}|$. (Here, $|X|$ denotes the number of elements in a set X.)
- Let V be a vector space. If V has a basis $\mathcal{B}$ with finitely many elements, then we say that V is finite-dimensional with dimension $|\mathcal{B}|$. As noted above, this number won't depend on the basis chosen.
- If V has no finite basis, then we say that V is infinite dimensional.
- If $\mathcal{B}$ is a basis of V, then every element of V can be written as a linear combination of the elements of $\mathcal{B}$ in a unique way.
- The dimension of $\mathbb{R}^n$ is n, since the standard basis $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ has n elements.
- A shortcut for checking whether something is a basis: if we know the dimension of V is n, then a subset $\mathcal{B}$ of V:
  - Can't be a basis of V unless $|\mathcal{B}| = n$
  - If $|\mathcal{B}| = n$, then to check whether it's a basis it's enough to check either that it's linearly independent or that it spans V. You don't need to check both.

17. Linear Transformations (Sections 5.1 and 5.2, what we've done of them.)

 - Let V and W be vector spaces. A function $T : V \to W$ is a linear transformation if for all $\vec{x}, \vec{y}$ in V and all scalars c,
   (1) $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$.
   (2) $T(c\vec{x}) = cT(\vec{x})$.
 - Checking whether functions are linear transformations
 - If T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, then there exists an $m \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$ for all vectors $\vec{x}$ in $\mathbb{R}^n$. Furthermore, the ith column of the matrix A is $T(\vec{e}_i)$.

18. Proofs, proofs, proofs!

 - Some general hints:
   - Write down what you're starting from and what you're proving
   - Making sure to start from the assumption, and work towards the conclusion
   - Explain your steps as if you had to explain what's going on to a classmate
 - Proof Techniques:
   - Proof by contrapositive: instead of proving that A implies B, prove that (not B) implies (not A).
   - Proof by induction: to show that some statement holds for any positive integer n (possibly for all n above a certain number), prove the following:
     (1) Base case: the statement holds for the smallest possible value of n, often $n = 1$ but not always.
     (2) Inductive step: if you assume that the statement holds for $n = k$, then it holds for $n = k + 1$.