In-Class Work Solutions for April 18th

Part 1:

1. Calculate the following limits. Only use L'Hospital's when appropriate.

   (a) $\lim_{x\to 0}(1+x)^{1/x}$.

   Solution:

   Attempting to plug in gives 1^∞, so we're going to need to do more work. Let's go through the algorithm.

   1. Let $y = (1+x)^{1/x}$. Then,
   $$\ln(y) = \ln\left((1+x)^{1/x}\right) = \frac{1}{x}\ln(1+x) = \frac{\ln(1+x)}{x}$$
   2. We need to calculate
   $$\lim_{x\to 0}\ln(y) = \lim_{x\to\infty}\frac{\ln(1+x)}{x}$$
   Note that
   $$\lim_{x\to 0}\ln(1+x) = \ln(1) = 0$$
   $$\lim_{x\to 0}x = 0$$
   so the form of the limit is $\frac{0}{0}$. Therefore, we can use L'Hospitals:
   $$\begin{aligned}\lim_{x\to\infty}\frac{\ln(1+x)}{x} &= \lim_{x\to\infty}\frac{1/(1+x)}{1}\\ &= \lim_{x\to\infty}\frac{1}{1+x} = 1\end{aligned}$$
   3. Using the result from Step 2, to get the actual limit we take e to the power of what we found. Thus,
   $$\boxed{\lim_{x\to 0}(1+x)^{1/x} = e^1 = e}$$

   (b) $\lim_{x\to 0}(1+\sin(x))^x$

   Solution:

   Here, it turns out we can just plug in.
   $$\boxed{\lim_{x\to 0}(1+\sin(x))^x = (1+\sin(0))^0 = 1^0 = 1}$$
   This works because 1^0 is not an indeterminate form – it's just 1.

Part 2:

1. Let $f(x) = \dfrac{e^x}{e^x - 1}$.

   (a) Find the horizontal asymptotes of $f(x)$. Feel free to use L'Hospital's when appropriate.

   Solution:

   To find horizontal asymptotes, take limits as x approaches ∞ and $-\infty$. Now, $\lim_{x\to\infty} \frac{e^x}{e^x-1}$ is clearly in the form $\frac{\infty}{\infty}$ since as $x \to \infty$, e^x approaches ∞. Therefore, we can use L'Hospital's:

$$\begin{aligned}\lim_{x\to\infty} \frac{e^x}{e^x - 1} &= \lim_{x\to\infty} \frac{(e^x)'}{(e^x - 1)'} \\ &= \lim_{x\to\infty} \frac{e^x}{e^x} = \lim_{x\to\infty} 1 = 1\end{aligned}$$

   Now, note that as $x \to -\infty$, e^x approaches 0. (Check the graph of e^x if you don't remember this!) Thus,

$$\begin{aligned}\lim_{x\to-\infty} \frac{e^x}{e^x - 1} &= \frac{\lim_{x\to-\infty} e^x}{\lim_{x\to-\infty}(e^x - 1)} \\ &= \frac{0}{0-1} = 0\end{aligned}$$

   Thus, $\boxed{y = 1}$ is the asymptote at ∞, and $\boxed{y = 0}$ is the asymptote at $-\infty$.

   (b) Find the vertical asymptotes of $f(x)$. For each vertical asymptote $x = L$, find

$$\lim_{x\to L^-} f(x) \text{ and } \lim_{x\to L^+} f(x)$$

   Solution:

   Vertical asymptotes exist where either the denominator is 0 or the numerator blows up. Since the numerator never blows up, set the denominator to 0:

$$\begin{aligned}&e^x - 1 = 0 \\ &\Rightarrow e^x = 1 \\ &\Rightarrow x = \ln(1) = 0\end{aligned}$$

   Thus, our only potential vertical asymptote is at $x = 0$. To see if it's actually an asymptote, check the left-hand and right-hand limits (one of them has to be $\pm\infty$, or it's not an asymptote):

$$\lim_{x\to 0^+} \frac{e^x}{e^x - 1} \approx \frac{e^{0.01}}{e^{0.01} - 1} \approx \frac{1}{\text{tiny positive \#}} = \text{big positive \#}$$

since e^x is an increasing function, and hence $e^{0.01}$ is greater than 1. Thus,

$$\boxed{\lim_{x\to 0^+} \frac{e^x}{e^x - 1} = \infty}$$

Similarly,

$$\lim_{x\to 0^-} \frac{e^x}{e^x - 1} \approx \frac{e^{-0.01}}{e^{-0.01} - 1} \approx \frac{1}{\text{tiny negative \#}} = \text{big negative \#}$$

and therefore

$$\boxed{\lim_{x\to 0^-} \frac{e^x}{e^x - 1} = -\infty}$$

The above calculations show that indeed, $\boxed{x = 0}$ is a vertical asymptote.

(c) Find the intervals on which $f(x)$ is increasing/decreasing.

Solution:

This, as usual, requires working with $f(x)$. Using the quotient rule,

$$\begin{aligned}
\left(\frac{e^x}{e^x - 1}\right)' &= \frac{(e^x - 1)(e^x)' - (e^x - 1)' e^x}{(e^x - 1)^2} \\
&= \frac{(e^x - 1)e^x - e^x \cdot e^x}{(e^x - 1)^2} \\
&= -\frac{e^x}{(e^x - 1)^2}
\end{aligned}$$

To find the places where $f'(x)$ could change sign, check where $f'(x) = 0$ or doesn't exist.

<u>$f'(x) = 0$</u>: For $f'(x)$ to be 0, the numerator must be 0. Thus,

$$e^x = 0$$

Since e^x is always positive, this has no solutions.

<u>$f'(x)$ doesn't exist</u>: This happens when the denominator is 0 or the numerator doesn't exist. Since the numerator always exists, set the denominator to 0:

$$\begin{aligned}
(e^x - 1)^2 &= 0 \\
\Rightarrow e^x &= 1 \\
\Rightarrow x &= \ln(1) = 0
\end{aligned}$$

Thus, the only places the derivative could change sign is at 0. Checking, we see that $f'(-1)$ is negative, as is $f'(1)$. Therefore, $f(x)$ is $\boxed{\text{decreasing on } (-\infty, 0) \cup (0, \infty)}$ (but it's not decreasing everywhere, since it's not defined at 0.)

(d) It turns out that the function is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. Use this, as well as the information gathered, to sketch $f(x)$.

Solution:

Here it is: