In-Class Work Solutions for April 2nd

1. Find the absolute maximum value of $f(x) = x(4-3x)^{1/3}$ on the interval $[0,4]$.

   Solution:

   Here, we have a continuous function on a closed interval. Thus, we can use the closed interval method. Let us go through the algorithm.

   1. We start by finding the critical points of f in $(0,4)$. To do so, begin by finding the derivative:

$$\begin{aligned}\left(x(4-3x)^{1/3}\right)' &= (x)'(4-3x)^{1/3} + x\left((4-3x)^{1/3}\right)' \\ &= 1\cdot(4-3x)^{1/3} + x\cdot\frac{1}{3}(4-3x)^{-2/3}\cdot(4-3x)' \\ &= (4-3x)^{1/3} - x(4-3x)^{-2/3}\end{aligned}$$

It now turns out to be easiest to combine the above two terms into a single fraction before proceeding:

$$\begin{aligned}f'(x) &= (4-3x)^{1/3} - x(4-3x)^{-2/3} \\ &= (4-3x)^{1/3} - \frac{x}{(4-3x)^{2/3}} \\ &= (4-3x)^{1/3}\cdot\frac{(4-3x)^{2/3}}{(4-3x)^{2/3}} - \frac{x}{(4-3x)^{2/3}} \\ &= \frac{(4-3x)^{1/3}(4-3x)^{2/3} - x}{(4-3x)^{2/3}} \\ &= \frac{(4-3x)-x}{(4-3x)^{2/3}} = \frac{4-4x}{(4-3x)^{2/3}}\end{aligned}$$

Now, we need to consider where $f'(x)$ doesn't exist and where $f'(x) = 0$.

<u>$f'(x)$ doesn't exist:</u> Here, we have a fraction whose denominator and numerator always exist. That means the only way $f'(x)$ doesn't exist is if the denominator is 0:

$$\begin{aligned}0 &= (4-3x)^{2/3} \\ \Rightarrow 4-3x &= 0 \\ \Rightarrow x &= \frac{4}{3}\end{aligned}$$

<u>$f'(x) = 0$:</u> This occurs when the numerator is 0: thus, we solve

$$\begin{aligned}4-4x &= 0 \\ \Rightarrow 4x &= 4 \\ \Rightarrow x &= 1\end{aligned}$$

Thus, we found two values of x where $f'(x)$ is either 0 or doesn't exist. Checking, we see that both are in the domain of f, and hence are actually critical numbers. Finally, we only take the critical numbers in the interval $(0, 4)$: however, in this case that includes both of them.

2. Now, we plug in the x-values from the previous step into f:

$$f(4/3) = 4/3 \cdot (4 - 3 \cdot 4/3)^{1/3} = 4/3 \cdot (0)^{1/3} = 0$$
$$f(1) = 1 \cdot (4 - 3 \cdot 1)^{1/3} = 1 \cdot 1^{1/3} = 1$$

3. Continue by plugging in the endpoints into f:

$$\begin{aligned} f(0) &= 0 \cdot (4 - 3 \cdot 0)^{1/3} = 0 \cdot 4^{1/3} = 0 \\ f(4) &= 4 \cdot (4 - 3 \cdot 4)^{1/3} = 4 \cdot (-8)^{1/3} \\ &= 4 \cdot (-2) = -8 \end{aligned}$$

4. Finally, comparing the values we got in Steps 2 and 3, we see that

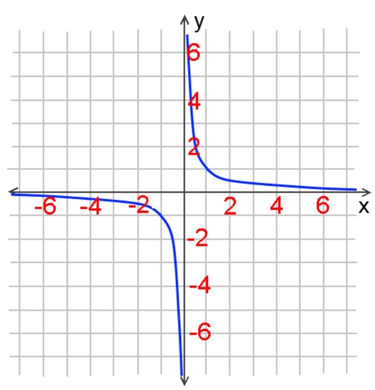
$$\boxed{\text{The absolute minimum of } f(x) \text{ is } -8, \text{ attained at } x = 4.}$$
$$\boxed{\text{The absolute maximum of } f(x) \text{ is } 1, \text{ attained at } x = 1.}$$

2. Consider the function $f(x) = \frac{1}{x}$.

(a) Sketch the function $f(x)$. What is its absolute minimum value on $[-1, 1]$?

Solution:

Clearly, $f(x)$ is a hyperbola. Here's a sketch:



It should be clear from the picture (and the fact that $f(x)$ approaches $-\infty$ as x approaches 0 from the left) that $f(x)$ doesn't have an absolute minimum on $[-1, 1]$.

(b) Try to use the closed interval method to find the absolute minimum of $f(x)$ on the interval $[-1, 1]$. What goes wrong?

Solution:

Let us try to go through the algorithm.

1. Start by finding the critical points of $f(x)$. Here,

$$f'(x) = \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

   Recall that a critical point is a number in the domain of f such that $f'(x)$ is 0 or doesn't exist. It's clear from the above formula that $f'(x)$ is never equal to 0, and that it doesn't exist when the denominator x^2 is 0. Thus, the only potential critical point is $x = 0$. However, since $f(0)$ isn't defined, 0 isn't in the domain of f and hence isn't actually a critical point. Therefore, there are no critical points.

2. We didn't find any critical points, so there's nothing to plug in.
3. Plug in the endpoints:

$$f(1) = \frac{1}{1} = 1$$
$$f(-1) = \frac{1}{-1} = -1$$

4. The closed interval test says that the absolute minimum value of $f(x)$ on $[-1, 1]$ is -1. This clearly doesn't match the picture!

What went wrong: In fact, it turns out that the closed interval test can't be used here! It can only be used for functions that are continuous on a given closed interval, and it should be abundantly clear from our picture that $f(x)$ isn't continuous at 0.