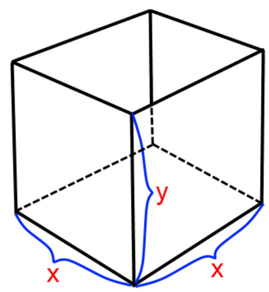
In-Class Work Solutions for April 23rd

1. Find the maximum volume of a box that can be constructed with 16 square feet of material, if its bottom must be square, and if its top is open.

   Solution:

   1. Here's the diagram:

   2. As labelled above, x is the length (and width) of the bottom of the box, and y is the height of the box. Also, let V be the volume of the box. We're maximizing V.
   3. $V = x^2y$, since the volume is just the product of the side lengths.
   4. Since we have 16 square feet of material, the total surface area must be 16. The area of the bottom of the box is x^2, and the area of each of the four sides is xy. Thus, the relationship is

   $$4xy + x^2 = 16$$

   5. Let us solve for one of the variables. Here, it's much easier to solve for y:

   $$\begin{aligned} 4xy + x^2 &= 16 \\ \Rightarrow 4xy &= 16 - x^2 \\ \Rightarrow y &= \frac{16 - x^2}{4x} \end{aligned}$$

   Thus, we get that

   $$A = x^2y = x^2 \cdot \frac{16 - x^2}{4x} = \frac{x(16 - x^2)}{4}$$

6. What values of x make sense in context? Since x is a side length, we must have $x \geq 0$. Furthermore, $x = 0$ would mean that the surface area was 0, which isn't consistent with the problem. (And it's easy to see that y would be undefined in that case.) Thus, we must have $x > 0$.

   Since we're using 16 square feet of material, the largest x could be is 4: in that case, the bottom of the box would already be 16 square feet, and the box would have no height at all. This corresponds to a degenerate box. Thus, we must have that $x \leq 4$.

   Therefore, the domain is $(0, 4]$.

7. Our task is to maximize $A(x) = \frac{x(16-x^2)}{4}$ over $(0, 4]$. Since we do not have a closed interval, the Closed Interval Test doesn't apply. Therefore, let us find the intervals of increase/decrease of the function. We have that

$$\begin{aligned} A'(x) &= \left(\frac{x(16-x^2)}{4}\right)' = \left(\frac{1}{4}(16x - x^3)\right)' \\ &= \frac{1}{4}(16 - 3x^2) \end{aligned}$$

   This is clearly always defined. Setting $A'(x) = 0$ gets

$$\begin{aligned} 0 &= \frac{1}{4}(16 - 3x^2) \\ \Rightarrow 3x^2 &= 16 \\ \Rightarrow x &= \pm\sqrt{\frac{16}{3}} = \pm\frac{4}{\sqrt{3}} \\ \Rightarrow x &= \frac{4}{\sqrt{3}} \end{aligned}$$

   since the other possibility isn't in the domain.

   Testing the sign of $A'(x)$ shows that $A'(x)$ is positive on $(0, 4/\sqrt{3})$ and negative on $(4/\sqrt{3}, 4]$. Thus, $A(x)$ is increasing on $(0, 4/\sqrt{3})$ and decreasing on $(4/\sqrt{3}, 4]$. This means that the absolute maximum is attained at $4/\sqrt{3}$, and it is

$$\begin{aligned} A(4/\sqrt{3}) &= \frac{4/\sqrt{3} \cdot \left(16 - \left(4/\sqrt{3}\right)^2\right)}{4} \\ &= \frac{16 - 16/3}{\sqrt{3}} = \frac{32/3}{\sqrt{3}} = \frac{32}{3\sqrt{3}} \end{aligned}$$

   Therefore,

$$\boxed{\text{The maximum volume of such a box is } \frac{32}{3\sqrt{3}} \text{ cubic feet}}$$

2. What is the maximum product of a number and 2 minus three times that number?

   1. No diagram needed.
   2. Let x be the number in question. Let P be the product we're asked about.
   3. Here, $P = x(2-3x)$.
   4. No relationships needed – P is already in terms of one variable.
   5. No further work needed.
   6. x is the number in question, and we're not told any constraints on this number. Thus, the domain is $(-\infty, \infty)$.
   7. We need to maximimize $P(x) = x(2-3x)$ over $(-\infty, \infty)$. Clearly, the Closed Interval Test doesn't apply, so we use intervals of increase/decrease. We have that

      $$P'(x) = (x(2-3x))' = \left(2x - 3x^2\right)' = 2 - 6x$$

      This is always defined, so set it equal to 0:

      $$\begin{aligned} 2 - 6x &= 0 \\ \Rightarrow 6x &= 2 \\ \Rightarrow x &= \frac{2}{6} = \frac{1}{3} \end{aligned}$$

      Testing the sign of $P'(x)$ yields that $P'(x)$ is positive on $(-\infty, 1/3)$ and negative on $(1/3, \infty)$. Thus, $P(x)$ increasing on $(-\infty, 1/3)$ and decreasing on $(1/3, \infty)$. Therefore, the absolute maximum must be attained at $1/3$, and is

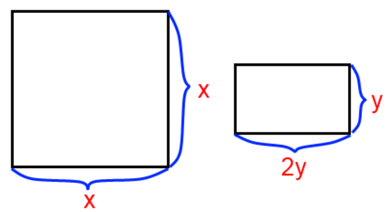
      $$\begin{aligned} P\left(\frac{1}{3}\right) &= \frac{1}{3} \cdot \left(2 - 3 \cdot \frac{1}{3}\right) = \frac{1}{3} \cdot (2-1) \\ &= \frac{1}{3} \end{aligned}$$

      Thus,

      $$\boxed{\text{The largest such product is } \frac{1}{3}}$$

3. We have a 10 foot wire. We split the wire into two pieces, and bend the first piece into a square, and the second piece into a rectangle one of whose sides is twice the other. What is the maximum combined area of the two shapes?

    1. Here's the diagram:

    2. As labelled above, x is the side length of the square, whereas y and $2y$ are the side lengths of the rectangle. Furthermore, let A be the combined area of the two shapes.
    3. $A = x^2 + 2y^2$.
    4. Since the total wire used for the shapes should be 10 feet, the total perimeter is 10 feet. The perimeter of the square is $4x$, and the perimeter of the rectangle is $6y$. Thus, the relationship is

    $$4x + 6y = 10$$

    5. Here, it doesn't matter which variable to solve for. Let's solve for y:

    $$\begin{aligned} 4x + 6y &= 10 \\ \Rightarrow 6y &= 10 - 4x \\ \Rightarrow y &= \frac{10 - 4x}{6} = \frac{5 - 2x}{3} \end{aligned}$$

    Thus, we get that

    $$A = x^2 + 2y^2 = x^2 + 2 \cdot \left(\frac{5 - 2x}{3}\right)^2$$

    6. What values of x make sense here? Obviously, the largest possible value of x corresponds to using all the wire for the square. This leads to $x = 10/4 = 5/2$. Furthermore, the smallest x could be is if all the wire is used for the other rectangle – in this case, $x = 0$. Therefore, the domain is $[0, 5/2]$.
    7. Here, we're maximizing

    $$A(x) = x^2 + 2 \cdot \left(\frac{5 - 2x}{3}\right)^2$$

over $[0, 5/2]$. Since we have a continuous function over a closed interval, the Closed Interval Test applies. Thus, we just need to find the critical points, and plug those critical points as well as the endpoints into the original function. We have that

$$\begin{aligned} A'(x) &= 2x + 2 \cdot \left(2 \cdot \frac{5-2x}{3} \cdot (-2)\right) \\ &= 2x - \frac{8(5-2x)}{3} = \frac{6x - 8(5-2x)}{3} \\ &= \frac{22x - 40}{3} \end{aligned}$$

This is clearly always defined. Setting it to 0 yields

$$\begin{aligned} 22x - 40 &= 0 \\ \Rightarrow 22x &= 40 \\ \Rightarrow x &= \frac{40}{22} = \frac{20}{11} \end{aligned}$$

Now, plugging in endpoints and critical points:

$$\begin{aligned} A(0) &= 0^2 + 2 \cdot \left(\frac{5 - 2 \cdot 0}{3}\right)^2 = 2 \cdot \frac{25}{9} = \frac{50}{9} \approx 5.56 \\ A(5/2) &= \left(\frac{5}{2}\right)^2 + 2 \cdot \left(\frac{5 - 2 \cdot (5/2)}{3}\right)^2 = \frac{25}{4} + 2 \cdot 0^2 = 6.25 \\ A(20/11) &= \left(\frac{20}{11}\right)^2 + 2 \cdot \left(\frac{5 - 2 \cdot (20/11)}{3}\right)^2 \approx 3.72 \end{aligned}$$

(Sorry, this definitely requires a calculator!) The biggest number clearly occurs at $5/2$. Therefore,

$$\boxed{\text{The maximum combined area of the two shapes is 6.25 square feet}}$$

Looking back, we see that the way to maximize area is to use all the wire for the square.

Optimization Problems

Algorithm for Optimization:

1. Draw a diagram (if appropriate.)
2. Label all the relevant variables. Make sure to give a name to the quantity being minimized or maximized. Note that with optimization, things are not functions of time!
3. Express this quantity in terms of the other variables.
4. Find relationships between the other variables enabling one to solve for everything in terms of one variable.
5. Use the expressions in the last step to express quantity being minimized or maximized in terms of one variable.
6. Find the domain of the function (this will depend on which values on the variables make sense in the context of the problem.)
7. Use methods from earlier (i.e. closed interval test, intervals of increase/decrease) to find the absolute min/max of our quantity on its domain.

The Closed Interval Test (Reminder): Let $f(x)$ be a continuous function on the closed interval $[a, b]$. Then, to find the absolute maximum and absolute minimum of $f(x)$ on $[a, b]$:

1. Find all the critical numbers of $f(x)$ that are in (a, b).
2. Plug in all the value from Step 1 into $f(x)$.
3. Plug in the endpoints a and b into $f(x)$.
4. The largest value found in Steps 2 and 3 is the absolute maximum of $f(x)$ on $[a, b]$; the smallest value found in Steps 2 and 3 is the absolute minimum of $f(x)$ on $[a, b]$.

Finding Absolute Mins/Maxes When Closed Interval Test Doesn't Apply:

1. Find the intervals of increase/decrease of $f(x)$.
2. Use logic to figure out where the absolute mins/maxes might be attained; test those points if there are more than one.