In-Class Questions for April 4th

Part 1:

1. Find the intervals on which $f(x) = x + \dfrac{1}{2x^2}$ is increasing and the intervals it's decreasing.

   Solution:

   Let us go through the algorithm.

   1. To find all places where $f'(x)$ could change sign, calculate $f'(x)$ and see where it's 0 or does not exist.

   $$\begin{aligned} f'(x) &= \left(x + \frac{1}{2x^2}\right)' = \left(x + \frac{1}{2}x^{-2}\right)' \\ &= 1 + \frac{1}{2}(-2)x^{-3} = 1 - x^{-3} \\ &= 1 - \frac{1}{x^3} = \frac{x^3 - 1}{x^3} \end{aligned}$$

   $f'(x) = 0$: To find where $f'(x)$ is 0, set the numerator to 0:

   $$\begin{aligned} x^3 - 1 &= 0 \\ \Rightarrow x^3 &= 1 \\ \Rightarrow x &= \sqrt[3]{1} = 1 \end{aligned}$$

   $f'(x)$ doesn't exist: To find where $f'(x)$ doesn't exist, set the denominator to 0:

   $$\begin{aligned} x^3 &= 0 \\ \Rightarrow x &= 0 \end{aligned}$$

   Thus, we have to plot the two points $x = 0$ and $x = 1$ and on our number line.

   2. We need to test the intervals $(-\infty, 0), (0, 1)$ and $(1, \infty)$. Let's plug in -1 for the first interval, 0.5 for the second interval, 2 for the third interval:

   $$\begin{aligned} f'(-1) &= \frac{(-1)^3 - 1}{(-1)^3} = \frac{-2}{-1} = 2 > 0 \\ f'(0.5) &= \frac{(0.5)^3 - 1}{(0.5)^3} = \frac{0.125 - 1}{0.125} = \frac{-0.875}{0.125} < 0 \\ f'(2) &= \frac{(2)^3 - 1}{2^3} = \frac{7}{8} > 0 \end{aligned}$$

Therefore, we see that

$$f'(x) \text{ is } \begin{cases} \text{positive} & x < 0 \\ \text{negative} & 0 < x < 1 \\ \text{positive} & 1 < x \end{cases}$$

3. Therefore, the answer is that $f(x)$ is $\boxed{\text{increasing on } (-\infty, 0) \text{ and } (1, \infty)}$, and $f(c)$ is $\boxed{\text{decreasing on } (0, 1)}$.

2. **The First Derivative Test:** Fill in the blanks in the following statement:

   If c is a critical number of a continuous function f, then

   - If $f'(x)$ changes from positive to negative at c, $f(x)$ has a <u>local max</u> at c.
   - If $f'(x)$ changes from negative to positive at c, $f(x)$ has a <u>local min</u> at c.

   If you're not sure, sketch a picture of $f(x)$ to see what's going on!

Part 2:

1. Let $f(x) = x + \dfrac{1}{2x^2}$, just like above.

   Use the answers from Part 1, the First Derivative Test, and common sense to determine:

   (a) The x-values at which $f(x)$ has local minimums and maximums.

   Solution:

   By the First Derivative Test, $f(x)$ has a local minimum at a critical number c if $f'(x)$ changes sign from negative to positive at c. We see from the answer to Problem 2 in Part 1 above that $f'(x)$ changes from negative to positive at $x = 1$. Furthermore, it is easy to check that 1 is in the domain of f, and such is actually a critical number. Therefore,

   $$\boxed{f(x) \text{ has a local minimum at } x = 1.}$$

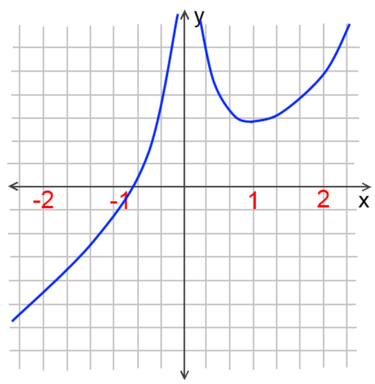
   Similarly, $f(x)$ has a local maximum at a critical number c if $f'(x)$ changes from positive to negative at c. The only place that $f'(x)$ changes from positive to negative is at $x = 0$. However, since $f(0)$ isn't defined, 0 isn't in the domain of f and as such isn't a critical number. Therefore, it's not a local max. Thus,

   $$\boxed{f(x) \text{ has no local maximums.}}$$

(b) The absolute minimum of $f(x)$ on $[-1, 1]$. Could we have also used the closed interval test?

Solution:

Looking at the answer from Part 1, Problem 2 we see that the shape of $f(x)$ (very approximately!) is something like this:



I drew the picture with an asymptote at 0 (which is indeed the case), but that's not even important for the logic of the problem. The important thing is just the shape.

Looking at the graph of the function, it's abundantly clear that the only possible places for an absolute minimum are at $x = 1$ and at $x = -1$: given the way it increases/decreases, every other point will be have a bigger y-value than one or the other of them. (I drew the picture as if it's smaller at -1, but again, that's not obvious without trying it!) Thus, we just need to test those points:

$$f(-1) = -1 + \frac{1}{2 \cdot (-1)^2} = -\frac{1}{2}$$
$$f(1) = 1 + \frac{1}{2 \cdot 1^2} = \frac{3}{2}$$

Therefore, we see that the absolute minimum on $[-1, 1]$ is attained at -1, and

$$\boxed{\text{The absolute minimum on } [-1, 1] \text{ is } -\frac{1}{2}.}$$

We **could not** have used the closed interval test – that only applies for functions that are continuous on a given interval, and our function isn't continuous at 0.

(c) The absolute minimum of $f(x)$ on $[0.5, 1.5]$. Could we have also used the closed interval test?

Solution:

Again, looking at the picture above, it's easy to see that the only possible place for an absolute minimum is at $x = 1$. (This is clear since it's bigger both to the right and to the left of $x = 1$.) Using the value we found by plugging in earlier, we see that

$$\boxed{\text{The absolute minimum on } [0.5, 1.5] \text{ is } \frac{3}{2}.}$$

We **could** have used the closed interval test, since our function is continuous on $[0.5, 1.5]$, as 0 is not included in that interval.