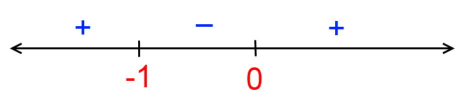
In-Class Work Solutions for April 9th

1. Let $f(x) = x^{2/3} + \dfrac{2x}{3}$, like last class. Recall that

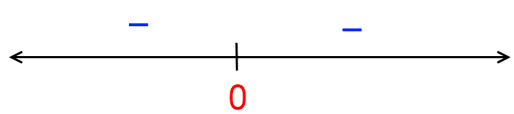
$$f'(x) = \frac{2}{3}x^{-1/3} + \frac{2}{3}$$
$$f''(x) = -\frac{2}{9}x^{-4/3}$$

The number line we found for f' was as follows:



As usual, the points on the above line were places were $f'(x)$ is 0 or doesn't exist.

The number line we found for f'' is here:



Very similarly to above, the places marked on this number line are the places where $f''(x)$ is 0 or doesn't exist.

Using all this information, do the following questions:

(a) Find the critical numbers for $f(x)$.

Solution:

As noted above, the numbers plotted on the number line for f' are where $f'(x)$ is 0 or doesn't exist. Thus, we just have to check which ones are actually in the domain of f. Rewriting,

$$f(x) = x^{2/3} + \frac{2x}{3} = (x^{1/3})^2 + \frac{2x}{3} = \sqrt[3]{x}^2 + \frac{2x}{3}$$

Since the cube root of x is defined for all x, and we don't have anything in the denominator, $f(x)$ is defined for all x. Therefore, the domain of x is all real numbers, and all the numbers on the f' number line are critical numbers. Thus,

$$\boxed{\text{The critical numbers for } f(x) \text{ are } -1 \text{ and } 0}$$

(b) Find the inflection points for $f(x)$.

Solution:

The inflection points are the points at which $f(x)$ changes concavity, as long as $f(x)$ is continuous there. Since looking at the number line for f'', $f(x)$ never changes concavity, there are no inflection points.

(c) Check whether each number from (a) is a local min or a local max using the first derivative test. Does this test apply to all of them?

Solution:

The first derivative test applies as long as the function is continuous at that point. Since our $f(x)$ is continuous everywhere, the first derivative test applies to all critical numbers.
At $x = -1$, $f'(x)$ changes from positive to negative. Thus,

$$\boxed{x = -1 \text{ is a local max.}}$$

At $x = 0$, $f'(x)$ changes from negative to positive. Thus,

$$\boxed{x = 0 \text{ is a local min.}}$$

(d) Check whether each number from (a) is a local min or a local max using the second derivative test. Does this test apply to all of them?

Solution:

The second derivative test only applies to critical numbers c where $f'(c)$ is equal to 0. Checking, we see that

$$\begin{aligned} &f'(0) \text{ doesn't exist} \\ &f'(-1) = 0 \end{aligned}$$

Therefore, the only critical number to which the second derivative test applies is $x = -1$. Plugging it into the second derivative,

$$\begin{aligned} f''(-1) &= -\frac{2}{9}(-1)^{-4/3} = -\frac{2}{9(-1)^{4/3}} \\ &= -\frac{2}{9(\sqrt[3]{-1})^4} = -\frac{2}{9(-1)^4} \\ &= -\frac{2}{9} \end{aligned}$$

Thus, the function is concave down at -1, and therefore $x = -1$ is a local max. This matches what we got with the first derivative test, which is good!

(e) Find the absolute minimum of $f(x)$ on $(-1, 1)$ by using the shape of the function. Does the Closed Interval Test apply?

Solution:

Our information for f' tells us that $f(x)$ is decreasing from -1 to 0 and then increasing from 0 to 1. (Sketch this if you need to visualize it!) This tells us that the only possible place for an absolute minimum is at $x = 0$. Since

$$f(0) = 0^{2/3} + \frac{2 \cdot 0}{9} = 0$$

we see that

$$\boxed{\text{The absolute minimum of } f(x) \text{ on } (-1, 1) \text{ is } 0\text{, attained at } 0}$$

The Closed Interval Test **does not** apply, since $(-1, 1)$ isn't a closed interval!

(f) Try to use the information about the shape of the graph above to sketch the function $f(x)$. If it helps, plot a couple of points!

Solution:

Here's the sketch: