22 FEB 2012

① Recall: The limit definition of the derivative as a function:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Given $f(x) = \cos(x)$, calculate $f'(x)$ using the limit definition of the derivative. The following formula and limits are useful:

① $\cos(x+h) = \cos(x)\cos(h) - \sin(x)\sin(h)$

② $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$

③ $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$

$$f(x) = \cos(x)$$

$$f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \cos(x) - \sin(x)\sin(h)}{h}$$

$$= \lim_{h\to 0} \left[\cos(x)\left(\frac{\cos(h)-1}{h}\right) - \sin(x)\left(\frac{\sin(h)}{h}\right)\right]$$

$$= \left[\lim_{h\to 0} \cos(x)\right]\left[\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right)\right] - \left[\lim_{h\to 0} \sin(x)\right]\left[\lim_{h\to 0} \frac{\sin(h)}{h}\right]$$

$$= \cos(x)(0) - \sin(x)(1)$$

$$f'(x) = \boxed{-\sin(x)}$$

Part 2:

① Calculate the derivative of $f(x) = \tan(x)$ using the quotient rule and the derivatives of $\sin(x)$ and $\cos(x)$.

$$f(x) = \tan(x)$$

$$= \frac{\sin(x)}{\cos(x)}.$$

$$f'(x) = \frac{\cos(x)[\sin(x)]' - \sin(x)[\cos(x)]'}{\cos^2(x)}$$

$$= \frac{\cos^2(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

Recall: $\sin^2(x) + \cos^2(x) = 1.$

$$= \frac{1}{\cos^2(x)}$$

Recall: $\frac{1}{\cos(x)} = \sec(x).$

$$f'(x) = \boxed{\sec^2(x)}$$

(4) Calculate the instantaneous rate of change of $f(x)$ at $x=0$, if $f(x) = x^2 \sin(x) + \cos(x)$.

Well, right from the beginning, we should apply the multiplication rule because x^2 and $\sin(x)$ are two terms that must be differentiated, yet they are being multiplied together:

$$f'(x) = x^2[\sin(x)]' + \sin(x)[x^2]' - \sin(x)$$

$$= x^2\cos(x) + 2x\sin(x) - \sin(x).$$

$$f'(0) = (0)^2\cos(0) + 2(0)\sin(0) - \sin(0)$$

$$= 0 + 0 + 0$$

$$f'(0) = \boxed{0}$$

Looks like this function is going nowhere fast. (This is the part where you laugh)