In-Class Work Solutions for February 29th

Part 1:

1. (a) Let $f(x) = 3^x$. Find $f'(x)$ using the chain rule the way just shown in class.

   Solution:

   We have that
   $$f(x) = 3^x = (e^{\ln(3)})^x = e^{\ln(3)x}$$
   Therefore, using the Chain Rule,
   $$\begin{aligned} f'(x) &= (e^{\ln(3)x})' = e^{\ln(3)x}(\ln(3)x)' \\ &= e^{\ln(3)x}\ln(3) \\ &= \boxed{3^x \ln(3)} \end{aligned}$$

   (b) Generalizing the example from lecture and part (a) above, formulate a rule for what the derivative of a^x is.

   Solution:

   We saw from class that the derivative of 2^x is $2^x \ln(2)$. Above, we saw that the derivative of 3^x is $3^x \ln(3)$. Generalizing, we see that
   $$\boxed{(a^x)' = a^x \ln(a)}$$

2. Find the instantaneous rate of change of $f(x)$ at $x = 1$, if
   $$f(x) = \frac{\sin(x^2 - 1)}{x - 2}$$

   Solution:

   The instantaneous rate of change of f at $x = 1$ is precisely $f'(1)$. Thus, we first find $f'(x)$ and then plug in $x = 1$.

   We start with the quotient rule:
   $$\begin{aligned} f'(x) &= \frac{(x-2)(\sin(x^2-1))' - \sin(x^2-1)(x-2)'}{(x-2)^2} \\ &= \frac{(x-2)(\sin(x^2-1))' - \sin(x^2-1)}{(x-2)^2} \end{aligned}$$

Now, we need to use the chain rule to find the derivatie of $\sin(x^2-1)$. Here, our inside function is $u = x^2 - 1$. Thus, we see that

$$\begin{aligned}(\sin(x^2-1))' &= (\sin(u))'u'(x) = \cos(u)\cdot(x^2-1)' \\ &= \cos(x^2-1)\cdot(2x) \\ &= 2x\cos(x^2-1)\end{aligned}$$

Plugging it back in, we see

$$\begin{aligned}f'(x) &= \frac{(x-2)(2x\cos(x^2-1)) - \sin(x^2-1)}{(x-2)^2} \\ &= \frac{2x(x-2)\cos(x^2-1) - \sin(x^2-1)}{(x-2)^2}\end{aligned}$$

Finally, we plug in $x = 1$:

$$\begin{aligned}f'(1) &= \frac{2\cdot 1\cdot(1-2)\cos(1^2-1) - \sin(1^2-1)}{(1-2)^2} \\ &= \frac{-2\cos(0) - \sin(0)}{1^2} = \frac{-2}{1} \\ &= \boxed{-2}\end{aligned}$$

Part 2:

1. Let $f(x) = \cos(e^x - 1)$.

   (a) Find the slope of the tangent to $f(x)$ at the point $(0, f(0))$.

   Solution:

   By definition, the slope of the tangent to $f(x)$ at $(0, f(0))$ is $f'(0)$. Thus, we need to differentiate $f(x)$ and then plug in 0.
   To differentiate, we use the chain rule. We have that $f(x) = \cos(e^x - 1)$, which can be rewritten as $\cos(u)$, where $u = e^x - 1$, which is our inside function. Thus,

   $$\begin{aligned}f'(x) &= (\cos(u))'u'(x) = (-\sin(u))(e^x-1)' \\ &= -\sin(e^x-1)e^x\end{aligned}$$

   Therefore,

   $$f'(0) = -\sin(e^0-1)e^0 = -\sin(1-1)\cdot 1 = -\sin(0)\cdot 1 = 0$$

   Thus, $\boxed{\text{the slope of the tangent to } f(x) \text{ at } (0, f(0)) \text{ is } 0.}$

(b) Use the result from part (a) to calculate

$$\lim_{h \to 0} \frac{\cos(e^h - 1) - 1}{h}$$

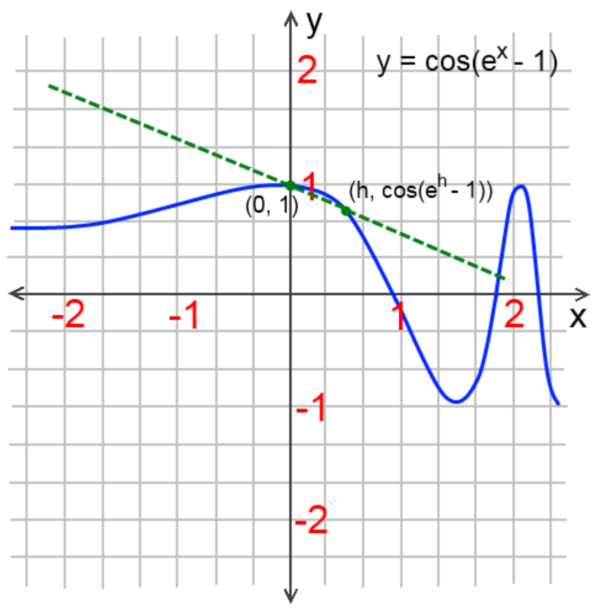
Hint: What geometric quantity does the above limit represent for the graph $y = \cos(e^x - 1)$?

Solution:

A little bit of thought shows that

$$\frac{\cos(e^h - 1) - 1}{h}$$

is the slope of the secant line connecting $(0, 1)$ and $(h, \cos(e^h - 1))$. Note that if $f(x) = \cos(e^x - 1)$, then these two points are $(0, f(0))$ and $(h, f(h))$. Here's a picture of this secant line:



We're taking the limit as h approaches 0: therefore, the secant line approaches the tangent line at $(0, 1)$. This means that

$$\lim_{h \to 0} \frac{\cos(e^h - 1) - 1}{h} = \text{Slope of tangent to } y = \cos(e^x - 1) \text{ at } (0, 1)$$

But we already calculated the above quantity in (a), since the answer to part (a) was indeed the slope of the tangent line to $y = \cos(e^x - 1)$

at $(0, f(0))$. Thus, the answer to part (b) is the same as to part (a): we have that

$$\boxed{\lim_{h \to 0} \frac{\cos(e^h - 1) - 1}{h} = 0}$$

Note: Another way to do this is to remember the formula

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

We see that if $f(x) = \cos(e^x - 1)$, and $x = 0$, then plugging in:

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{\cos(e^h - 1) - 1}{h} \end{aligned}$$

Thus, we would again be able to conclude that the answer to part (b) is the same as to part (a).