In-Class Work Solutions for February 3rd

Part 1:

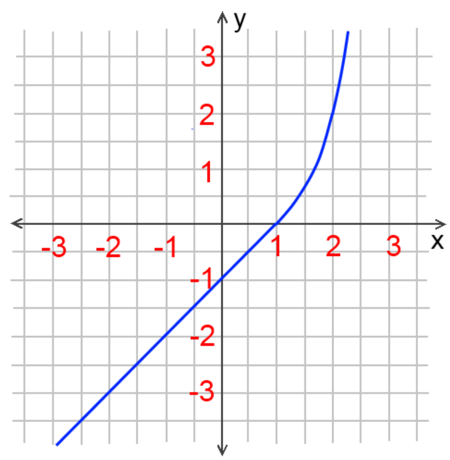
1. At which points is the function

$$f(x) = \begin{cases} x - 1 & x \leq 1 \\ x^2 - x & x > 1 \end{cases}$$

discontinuous? Make sure you can do the question both with a graph and the limit calculation!

Solution:

The graph of this function looks like:



This implies that $\boxed{\text{the function is continuous everywhere.}}$

Now, let's do this question without the picture. Since the two pieces of the function are polynomials, they are continuous everywhere. Therefore, the only possible discontinuity is at $x = 1$. To check whether a discontinuity occurs at 1, we need to do two things:

(a) Check that the limit at 1 exists using left-hand and right-hand limits.

(b) If the limit exists, check if it's equal to $f(1)$.

Let's first check whether the limit exists. We have that

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x - 1) = 1 - 1 = 0$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} (x^2 - x) = 1^2 - 1 = 0$$

Therefore, since the two one-sided limits match, we have that

$$\lim_{x \to 1} f(x) = 0$$

Now, we need to see if the limit is equal to $f(1)$. By definition, for $x \leq 1$, $f(x) = x - 1$. Thus, $f(1) = 1 - 1 = 0$, which means that

$$\lim_{x \to 1} f(x) = f(1)$$

Therefore, $f(x)$ is continuous at 1 by definition.

2. If $f(x) = x^2$ for all $x \neq 3$, and f is continuous at 3, what's $f(3)$?
Hint: If it helps, try to graph $f(x)$!

Solution:

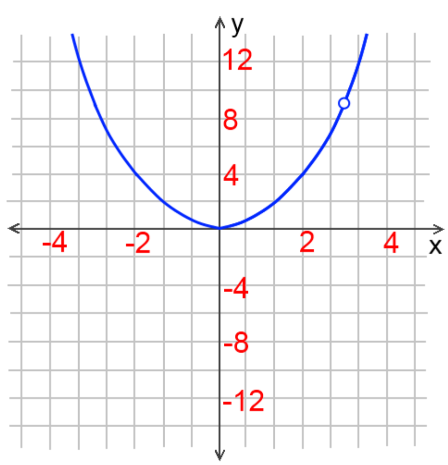
This question can be done with a simple limit calculation. However, let us first explain what's going on.

In this questions, we're given two pieces of information about $f(x)$:

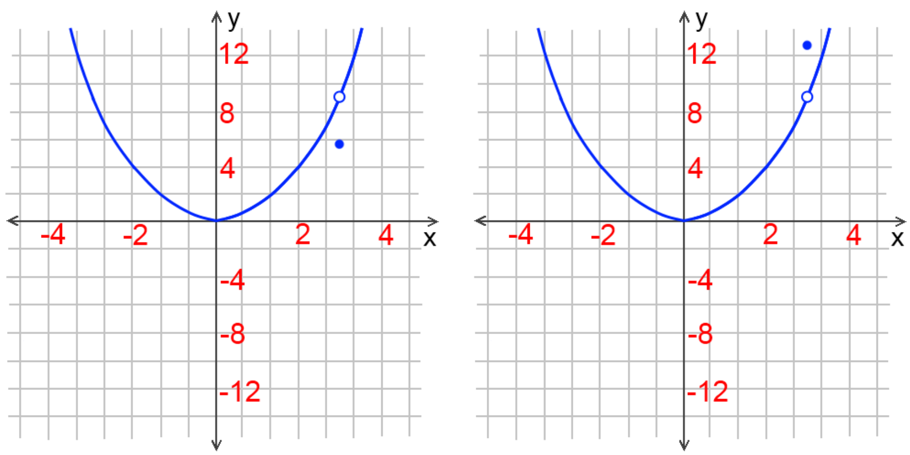
(a) It's equal to x^2 for all $x \neq 3$.

(b) It's continuous at 3.

If we start by using the first piece of information to graph $f(x)$, here's what we get:

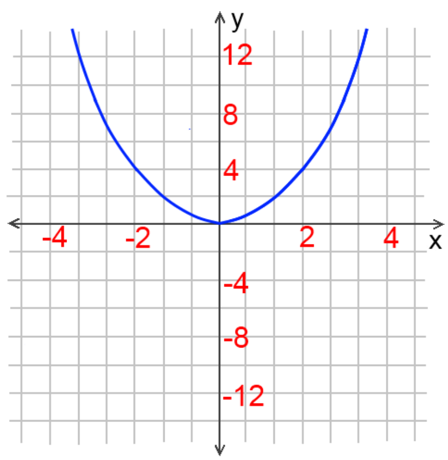


However, this is not the graph of $f(x)$: this is just the piece of the graph we get from the first bit of information. Without any more information, we wouldn't know anything at all about the value of f at 3. Thus, there'd be numerous possibilities for the actual graph of f: here are some other graphs other the one above that satisfy $f(x) = x^2$ for all $x \neq 3$:



That is, we need to somehow fill out a closed circle to for the y-value corresponding to $x = 3$ (or we could just leave it open).

However, since we are given that $f(x)$ is continuous at 3, we see that the only way to accomplish that is to fill in the closed circle. This means that the graph of $f(x)$ must be:



This shows us that

$$\boxed{f(3) = 9}$$

Now that we see how this works, let's do the quick limit calculation that shows it. Since f is continuous at 3, we have that

$$f(3) = \lim_{x \to 3} f(x)$$

Since the limit by definition doesn't care what happens at 3, and $f(x)$ is x^2 everywhere but 3, we see that

$$f(3) = \lim_{x \to 3} x^2 = 3^2 = 9$$

getting the same result as before.

Part 2:

1. For which values of x is the function

$$f(x) = \frac{x-3}{x^2+3x+2}$$

continuous? Write your answer in interval notation.

Solution:

This function is a quotient of two polynomials. Since polynomials are continuous everywhere, the only discontinuities will happen when the denominator is 0. Thus, we solve

$$\begin{aligned} x^2 + 3x + 2 &= 0 \\ \Rightarrow (x+2)(x+1) &= 0 \\ \Rightarrow x &= -2, -1 \end{aligned}$$

Therefore, the function is continous on all remaining real numbers. In interval notation that is written as

$$\boxed{(-\infty, -2) \cup (-2, -1) \cup (-1, \infty)}$$

2. At which points is the function

$$f(x) = \begin{cases} \frac{x^2+x-2}{x^2-3x+2} & x \neq 1 \\ -\frac{3}{2} & x = 1 \end{cases}$$

continuous? Again, answer should be in interval notation.
Hint: Make sure you know how to factor quadratics!

Solution:

This function is defined as $\frac{x^2+x-2}{x^2-3x+2}$ everywhere but at $x = 1$. Thus, everywhere but at $x = 1$, it suffices to consider where $\frac{x^2+x-2}{x^2-3x+2}$ is continuous. As this is again a rational function, we find discontinuities by setting the denominator to 0. We get

$$\begin{aligned} x^2 - 3x + 2 &= 0 \\ \Rightarrow (x-1)(x-2) &= 0 \\ \Rightarrow x &= 1, 2 \end{aligned}$$

Thus, $f(x)$ has a discontinuity at 2. However, we still need to check $x = 1$, since $f(1)$ is defined separately to be $-\frac{3}{2}$. We need to check whether $\lim_{x\to 1} f(x) = f(1)$. Working it out,

$$\begin{aligned}\lim_{x\to 1} f(x) &= \lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - 3x + 2} = \lim_{x\to 1} \frac{(x+2)(x-1)}{(x-2)(x-1)} \\ &= \lim_{x\to 1} \frac{x+2}{x-2} = \frac{3}{-1} = -3\end{aligned}$$

Therefore, we see that since $f(1) = -\frac{3}{2}$,

$$f(1) \neq \lim_{x\to 1} f(x)$$

and therefore $f(x)$ is discontinuous at 1 as well. Thus, $f(x)$ is continuous on

$$\boxed{(-\infty, 1) \cup (1, 2) \cup (2, \infty)}$$