In-Class Work Solutions for February 8th

Part 1:

1. Using the same kind of geometric reasoning we did in class, specify what the following quantities mean. (For example, the answer could be "the slope of the secant line connecting $(2, f(2))$ to $(3, f(3))$").

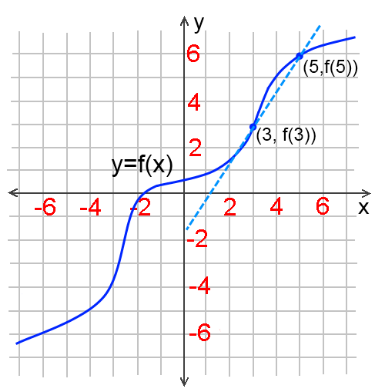
   If your answer is the slope of a tangent line, make sure to also express it as a derivative!

   (a)
   $$\frac{f(3+r)-f(3)}{r}$$
   where $r = 2$.

   Solution:

   We can start by plugging in $r = 2$. We get the expression
   $$\frac{f(5)-f(3)}{2}$$
   Rearranging, we see that this is equal to $\frac{f(5)-f(3)}{5-3}$, which looks suspiciously like the formula for slope. Indeed, this quantity is equal to $\boxed{\text{the slope of the secant connecting } (3, f(3)) \text{ and } (5, f(5)).}$

   Here's a picture reflecting what's going on. Note that the shape of the curve $y = f(x)$ doesn't matter at all for these questions – we're trying to visually represent what's going on for a general function $f(x)$.



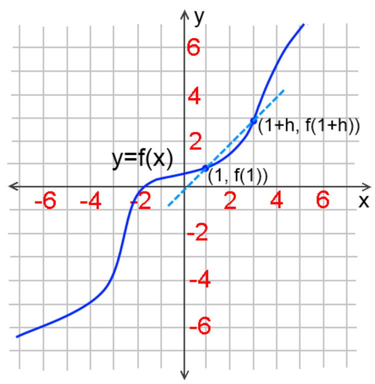
   As we can see from the picture, the rise for this secant line is $f(5) - f(3)$, while the run is $5 - 3 = 2$. Thus, its slope is precisely the intended expression.

(b)

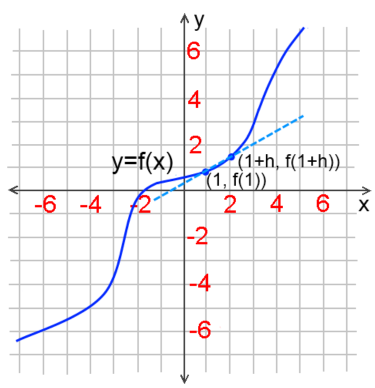
$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

Solution:

Let us first consider the quantity inside the limit, which is $\frac{f(1+h)-f(1)}{h}$. This is the slope of the secant line connecting $(1, f(1))$ to $(1+h, f(1+h))$. Here's a picture of that line (for some value of h – obviously this line varies with h):

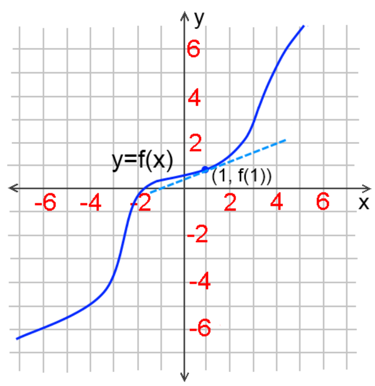


Now, we need to take the limit as $h \to 0$. This means we let h get smaller and smaller. Here's a picture of the secant line for a smaller value of h:



As h gets smaller, we see that the two points get closer and closer together. Therefore, the secant line gets closer and closer to the tangent line at $(1, f(1))$. Hence, the limit of the slopes of the secants as $h \to 0$ is precisely the slope of the tangent.

Here's a picture of that line:



Thus, the answer is

$$\boxed{\text{the slope of the tangent line at } (1, f(1))}$$

This quantity is also known as $\boxed{f'(1)}$.

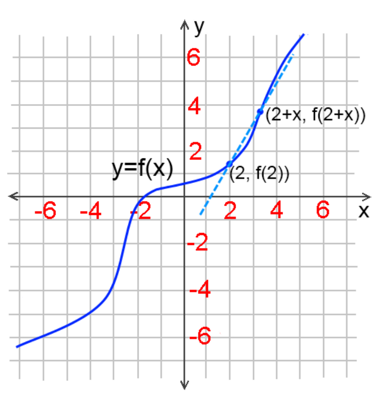
(c)

$$\lim_{x \to 1} \frac{f(2+x) - f(2)}{x}$$

Careful: this one is tricky! Make sure to draw the picture.

Solution:

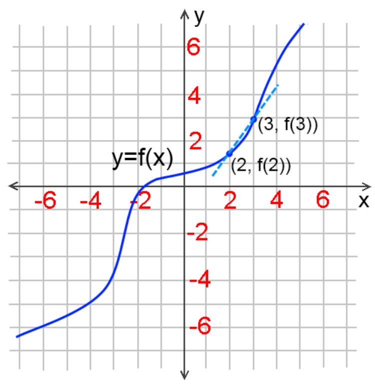
Here, we're again taking a limit of slopes of secants: in this question, the secants under discussion connect $(2, f(2))$ and $(2+x, f(2+x))$. Here's a picture of such a secant:



Since we're letting x approach 1, reasoning much like in part (b), we see that the secant lines get closer and closer to the secant line connecting $(2, f(2))$ and $(3, f(3))$. Thus, the slopes get closer to that slope as well, so the answer is

$$\boxed{\text{the slope of the secant line connecting } (2, f(2)) \text{ and } (3, f(3))}$$

Here's a picture of that line:

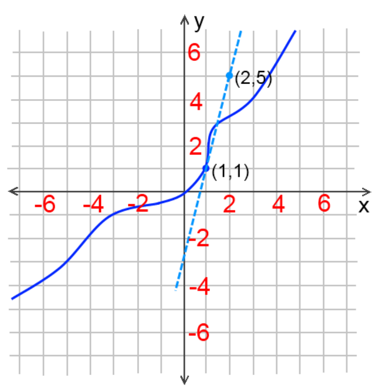


Part 2:

1. Assume that the tangent line to the curve $y = f(x)$ at $(1, 1)$ goes through the point $(2, 5)$. Calculate $f'(1)$.

   Solution:

   This question is easiest starting with a picture:



We're looking for $f'(1)$, which is by definition the slope of the tangent to $y = f(x)$ at the point $(1, f(1))$. Since we know that the curve contains the point $(1, 1)$, this means we're looking for the slope of the tangent line at $(1, 1)$. But we're told that this tangent line goes through the point $(2, 5)$. Summing up the logic, here's what we get:

$$\begin{aligned} f'(1) &= \text{slope of the tangent at } (1, f(1)) \\ &= \text{slope of the tangent at } (1, 1) \\ &= \frac{5-1}{2-1} = 4 \end{aligned}$$

Therefore, $\boxed{f'(1) = 4}$.

2. Let $f(t)$ denote a person's height at age t, measured in years. Would you expect $f'(2)$ to be positive or negative, and why? How about $f'(70)$?

   Solution:

   We know that $f'(t)$ is the rate of change of the function at time t. This means that $f'(a)$ is positive if $f(t)$ is increasing around time a, and $f'(a)$ is negative if $f(t)$ is decreasing around time a.

   At age 2, people's height is generally increasing. Thus, we see that $f(t)$ is increasing around time 2, and therefore we expect $f'(2)$ to be positive:

$$\boxed{f'(2) > 0}$$

   At age 70, people's height is expected to decrease. Thus, $f'(70)$ should be negative:

$$\boxed{f'(70) < 0}$$

   (If you believe that people's height isn't changing at all at age 70, then you could argue that $f'(70) = 0$. Either answer is fine with a good justification!)

   Note: If this logic is difficult, try to draw a graph of $f(t)$ – that is, of a person's age versus time – and take a look at the slopes!