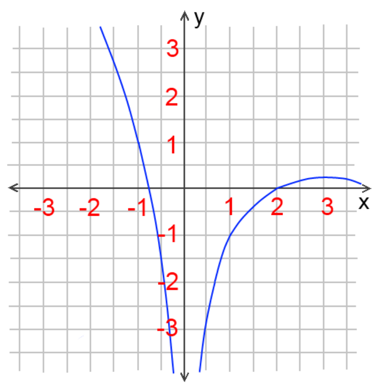
In-Class Questions for March 19th

1. A point is moving along the curve $2^{x+y} = x^2$. When the point is at $(1, -1)$, the x-coordinate is moving to the left at the speed of 2 units per second. What is the speed of the y-coordinate, and which direction is it going? (You may need a calculator to find the direction.)

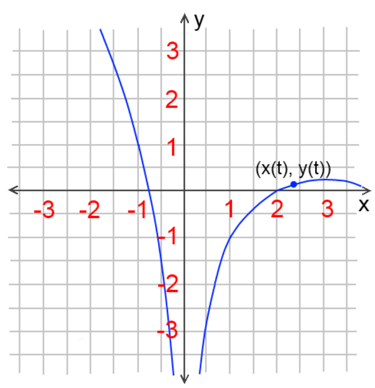
   Note: You don't really need this, but here's an approximate sketch of the curve:



Solution:

Let us follow the algorithm.

1. **Draw a diagram:**



2. **Label the variables:** in the above picture, $x(t)$ is the x-coordinate of the point at time t, and $y(t)$ is the y-coordinate of the point at time t. (We label these as functions of t since they are all changing with time.)

3. **Write down information given using derivatives:** we are given that when the point is at $(1,-1)$, the x-coordinate is moving to the left at the rate of 2 units per second. The rate of change of the x-coordinate is by definition $x'(t)$; since it's moving to the left, we know that it's decreasing. Thus, we're given that

$$x'(t) = -2 \text{ when } (x(t), y(t)) = (1, -1)$$

4. **Write down what we want to find using derivatives:** we're looking for how quickly the y-coordinate is changing when the point is at $(1,-1)$. By definition, the rate of change of the y-coordinate is $y'(t)$; therefore, we're asked

$$\text{What is } y'(t) \text{ when } (x(t), y(t)) = (1, -1)?$$

5. **Find a relationship**: By definition, if a point $(x(t), y(t))$ is on the curve $2^{x+y} = x^2$, then it satisfies the equation of the curve. Therefore, the relationship is very easy: it's just

$$2^{x(t)+y(t)} = x(t)^2$$

6. **Differentiate both sides of relationship with respect to t:** differentiating, (not forgetting the chain rule), we get

$$\begin{aligned} \left(2^{x(t)+y(t)}\right)' &= (x(t)^2)' \\ \Rightarrow \ln(2)2^{x(t)+y(t)}(x'(t) + y'(t)) &= 2x(t)x'(t) \end{aligned}$$

Now, to solve for $y'(t)$ (which is our required quantity), we need to first expand things out, then put all the terms with $y'(t)$ on one side and all the other terms on the other side:

$$\begin{aligned} \ln(2)2^{x(t)+y(t)}x'(t) + \ln(2)2^{x(t)+y(t)}y'(t) &= 2x(t)x'(t) \\ \Rightarrow \ln(2)2^{x(t)+y(t)}y'(t) &= 2x(t)x'(t) - \ln(2)2^{x(t)+y(t)}x'(t) \\ \Rightarrow y'(t) &= \frac{2x(t)x'(t) - \ln(2)2^{x(t)+y(t)}x'(t)}{2^{x(t)+y(t)}\ln(2)} \end{aligned}$$

7. **Substitute information given:** we're asked about $y'(t)$ when $(x(t), y(t)) = (1, -1)$. We also know that $x'(t)$ at this point is -2. Plugging all this into the above equation, we get that

$$\begin{aligned} y'(t) &= \frac{2x(t)x'(t) - \ln(2)2^{x(t)+y(t)}x'(t)}{2^{x(t)+y(t)}\ln(2)} \\ &= \frac{2 \cdot 1 \cdot (-2) - \ln(2) \cdot 2^{1+(-1)} \cdot (-2)}{2^{1+(-1)}\ln(2)} \\ &= \frac{-4 + 2\ln(2)}{\ln(2)} \approx -3.7708 \end{aligned}$$

Note that the answer is negative, and therefore the y-coordinate is decreasing. (This should also be clear from the graph!)

Therefore, the final answer is that when the point is at $(1, -1)$, the y-coordinate is decreasing at the rate of

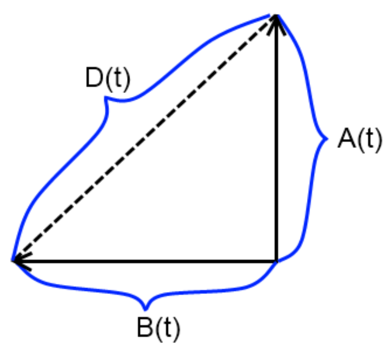
$$\boxed{\frac{4 - 2\ln(2)}{\ln(2)} \approx 3.7708 \text{ units per second.}}$$

2. Two ships A and B start off at the same point at noon. Ship A travels north at 30 miles/hour while ship B travels west at 40 miles/hour. How fast is the distance between the ships increasing at 2 pm?

   Solution:

   Let us follow the algorithm.

   1. **Draw a diagram:**



   2. **Label the variables:** in the above picture, $A(t)$ is the distance ship A has travelled after time t, and $B(t)$ is the distance ship B has travelled after time t. $D(t)$ is the distance between the ships at time t.
   3. **Write down information given using derivatives:** we are given that ship A travels at 30 miles per hour, and ship B travels at 40 miles per hour. Therefore, we're given that

$$A'(t) = 30$$
$$B'(t) = 40$$

   4. **Write down what we want to find using derivatives:** we're looking for how quickly the distance is changing at 2 pm. Thefore, the quesiton is:

      What is $D'(t)$ at 2 pm?

5. **Find a relationship**: it's clear that above we have a right triangle. Therefore, Pythagoras gives us that

$$A(t)^2 + B(t)^2 = D(t)^2$$

6. **Differentiate both sides of relationship with respect to t:** differentiating, (not forgetting the chain rule), we get

$$\begin{aligned}(A(t)^2 + B(t)^2)' &= (D(t)^2)' \\ \Rightarrow 2A(t)A'(t) + 2B(t)B'(t) &= 2D(t)D'(t)\end{aligned}$$

   Now, to solve for $D'(t)$ (which is our required quantity), we need to first expand things out, we simply divide, getting:

$$\begin{aligned}D'(t) &= \frac{2A(t)A'(t) + 2B(t)B'(t)}{2D(t)} \\ &= \frac{A(t)A'(t) + B(t)B'(t)}{D(t)}\end{aligned}$$

7. **Substitute information given:** we're asked about $D'(t)$ at 2 pm. Since ship A travels at 30 miles per hour, at 2 pm, $A(t) = 2 \cdot 30 = 60$ miles, and similarly $B(t) = 2 \cdot 40 = 80$ miles. Furthermore, using the relationship between the variables, we see that at 2 pm,

$$\begin{aligned}D(t)^2 &= A(t)^2 + B(t)^2 = 60^2 + 80^2 \\ &= 3600 + 6400 = 10,000 \\ \Rightarrow D(t) &= \sqrt{10,000} = 100\end{aligned}$$

   Therefore, plugging in, we have that

$$\begin{aligned}D'(t) &= \frac{A(t)A'(t) + B(t)B'(t)}{D(t)} \\ &= \frac{60 \cdot 30 + 80 \cdot 40}{100} \\ &= \frac{1800 + 3200}{100} = \frac{5000}{100} \\ &= 50\end{aligned}$$

   Therefore, the final answer is that

At 2 pm, the distance is increasing at 50 miles per hour.