In-Class Work Solutions for March 5th

Part 1:

1. (a) Calculate the derivative of $\log_2(x)$ using the method just used in class.

   Solution:

   $\log_2(x)$ is inverse function of 2^x. Therefore, following the steps, we get:

$$y = \log_2(x)$$
$$\Rightarrow 2^y = 2^{\log_2(x)} = x$$

   Now, we take the derivative of both sides with respect to x; we have to use implicit differentiation for 2^y. Don't forget, the derivative of 2^x is $2^x \ln(2)$.

$$(2^y)' = (x)'$$
$$\Rightarrow 2^y \ln(2) y' = 1$$

   Now, solving for y' we get

$$y' = \frac{1}{2^y \ln(2)}$$

   Finally, substituting the expression $y = \log_2(x)$, we get

$$y' = \frac{1}{2^{\log_2(x)} \ln(2)} = \boxed{\frac{1}{x \ln(2)}}$$

   (b) Formulate a rule for the derivative of $\log_a(x)$.

   Solution:

   Generalizing from part (a), we see that

$$\boxed{\log_a(x)' = \frac{1}{x \ln(a)}}$$

2. Calculate $f'(x)$, if

$$f(x) = \ln(1 + 4x^2) + \tan^{-1}(2x)$$

   Simplify your answer into a single fraction.

Solution:

Here, we have to use the chain rule separately on the two pieces:

$$\begin{aligned} f'(x) &= (\ln(1+4x^2) + \tan^{-1}(2x))' \\ &= (\ln(1+4x^2))' + (\tan^{-1}(2x))' \\ &= \frac{1}{1+4x^2} \cdot (1+4x^2)' + \frac{1}{1+(2x)^2} \cdot (2x)' \\ &= \frac{8x}{1+4x^2} + \frac{2}{1+4x^2} = \boxed{\frac{8x+2}{1+4x^2}} \end{aligned}$$

3. Calculate the derivatives of the following functions using logarithmic differentiation. If it can be done *without* logarithmic differentiation, check your answer by differentiating in the normal way.

   (a) $f(x) = e^{x^2}$.

   Solution:

   Using logarithmic differentation, we start by writing

   $$y = e^{x^2}$$

   We then take ln of both sides and simplify:

   $$\ln(y) = \ln\left(e^{x^2}\right) = x^2$$

   Now, differentiate both sides with respect to x; we need to use implicit differentiation for $\ln(y)$:

   $$\begin{aligned} (\ln(y))' &= (x^2)' \\ \Rightarrow \frac{1}{y}y' &= 2x \end{aligned}$$

   Finally, solving for y' and substituting in the formula for y:

   $$y' = 2xy = \boxed{2xe^{x^2}}$$

   Now, note that this question can also be quickly done with chain rule:

   $$(e^{x^2})' = (e^{x^2}) \cdot (x^2)' = e^{x^2} \cdot 2x = 2xe^{x^2}$$

   Clearly, we get the same answer either way!

   (b) $f(x) = x^{2\sin(x)}$.

Solution:

Following the exactly same algorithm as above:

$$y = x^{2\sin(x)}$$
$$\Rightarrow \ln(y) = \ln(x^{2\sin(x)}) = 2\sin(x)\ln(x)$$

Now, differentiating both sides with respect to x:

$$(\ln(y))' = (2\sin(x)\ln(x))'$$
$$\Rightarrow \frac{1}{y}y' = 2\sin(x)(\ln(x))' + 2(\sin(x))'\ln(x)$$
$$= 2\frac{\sin(x)}{x} + 2\cos(x)\ln(x)$$

Solving for y', and substituting the expression for y, we get

$$y' = y\left(2\frac{\sin(x)}{x} + 2\cos(x)\ln(x)\right)$$
$$= \boxed{x^{2\sin(x)}\left(2\frac{\sin(x)}{x} + 2\cos(x)\ln(x)\right)}$$