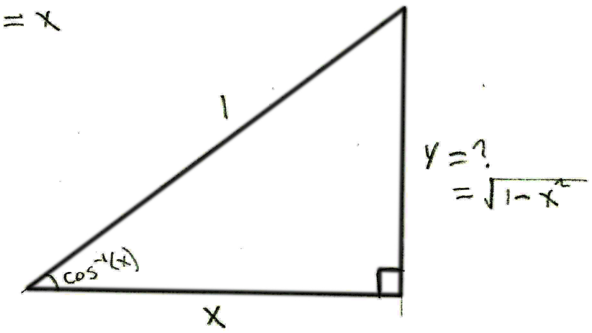
GROUP WORK
7 MAR

PART #1

a) Recall: $(\cos^{-1}(x))' = \frac{-1}{\sqrt{1-x^2}}$, try it for yourself.

Now we try to find it using the algorithm:

1) $y = \cos^{-1}(x)$
2) $\cos(y) = \cos(\cos^{-1}(x)) = x$
   $\cos(y) = x$
3) $(\cos(y))' = (x)'$
   $-\sin(y) \cdot y' = 1$
   $y' = \frac{-1}{\sin(y)}$
4) $y' = \frac{-1}{\sin(\cos^{-1}(x))}$



I use the identity

$$\cos(\cos^{-1}(x)) = \frac{x}{1}$$

to construct a triangle with the adjacent side's length being equal to x, and the hypotenuse side's length equal to 1. We can find the remaining side length by using the all too familiar Pythagorean Theorem, $a^2 + b^2 = c^2$.

$x^2 + y^2 = 1^2$
$y^2 = 1 - x^2$
$y = \sqrt{1-x^2}$.

Remember, $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$, and our angle, θ, is equal to $\cos^{-1}(x)$, so we can extract information from our completed triangle diagram to find

$$\sin(\cos^{-1}(x)) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.$$

Looking back to part 4 of the algorithm, we see that we have found a way to make our result,

$$y' = \frac{-1}{\sin(\cos^{-1}(x))},$$

look more like the identity by substituting $\sqrt{1-x^2}$ for $\sin(\cos^{-1}(x))$.

$$\boxed{y' = \frac{-1}{\sin(\cos^{-1}(x))} = \frac{-1}{\sqrt{1-x^2}}.}$$

And, we can construct triangles for the following trig identities as well:

$\sin(\sin^{-1}(x)) = x$
$\tan(\tan^{-1}(x)) = x$.

b) Recall: $\left(\tan^{-1}(x)\right)' = \dfrac{1}{1+x^2}$.

Guess what? More implicit differentiation fun!

1) $y = \tan^{-1}(x)$

2) $\tan(y) = \tan(\tan^{-1}(x)) = x$

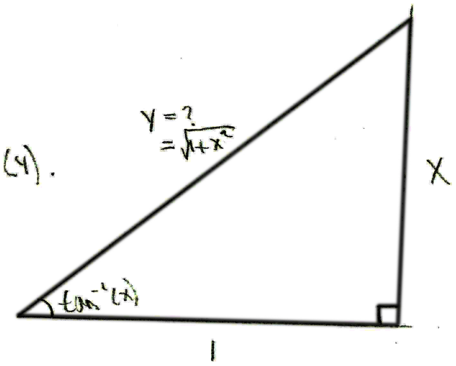
3) $(\tan(y))' = (x)'$

$\sec^2(y) \cdot y' = 1$

$y' = \dfrac{1}{\sec^2(y)} = \cos^2(y)$.

4) $y' = \cos^2(y) = \cos^2(\tan^{-1}(x))$.

$y' = \left[\cos(\tan^{-1}(x))\right]^2$.



Use the identity

$$\tan(\tan^{-1}(x)) = \frac{x}{1}$$

to construct a triangle with an angle $\theta = \tan^{-1}(x)$. Use Pythagorean's Theorem to find the length of the hypotenuse in terms of x.

$y^2 = 1^2 + x^2$

$y = \sqrt{1+x^2}$.

Remember, $\cos(\theta) = \dfrac{\text{adjacent}}{\text{hypotenuse}}$, and our angle, θ, is equal to $\tan^{-1}(x)$, so we can extract information from our constructed triangle diagram to find

$$\cos(\tan^{-1}(x)) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{1+x^2}}.$$

Looking back to part 4 of the algorithm, we see that we have found a way to make our result,

$$y' = \cos^2(\tan^{-1}(x)) = \left[\cos(\tan^{-1}(x))\right]^2,$$

look more like the identity by substituting $\dfrac{1}{\sqrt{1+x^2}}$ for $\cos(\tan^{-1}(x))$.

$$\boxed{y' = \cos^2(\tan^{-1}(x)) = \left[\cos(\tan^{-1}(x))\right]^2 = \left[\frac{1}{\sqrt{1+x^2}}\right]^2 = \frac{1}{1+x^2}}$$

PART #2

a) Normal Differentiation? $f(x) = [\ln(x)]^x$

$$f(x) = [\ln(x)]^x$$

$$f'(x) = \frac{d}{dx}[u^v] = \frac{du^v}{du}\cdot\frac{du}{dx} + \frac{du^v}{dv}\cdot\frac{dv}{dx}, \quad u = \ln(x);\ v = x.$$

I strongly urge all of you to avoid this and use logarithm differentiation instead.

$$= (u^{v-1}\cdot v)\cdot(u)' + (u^v\cdot\ln(u))\cdot(1)$$

$$= [\ln(x)]^{x-1}(x)\left(\frac{1}{x}\right) + [\ln(x)]^x\cdot\ln(\ln(x))\cdot(1)$$

$$= \frac{[\ln(x)]^x}{[\ln(x)]} + [\ln(x)]^x\ln(\ln(x))$$

$$= \boxed{[\ln(x)]^x\left(\ln(\ln(x)) + \frac{1}{\ln(x)}\right)}$$

Implicit:

$$f(x) = [\ln(x)]^x$$

$$y = [\ln(x)]^x$$

ln of both sides $\quad \ln(y) = \ln([\ln(x)]^x)$

log-Rule $\quad \ln(y) = x\ln(\ln(x))$

$$[\ln(y)]' = (x\cdot\ln(\ln(x)))'$$

$$\frac{1}{y}\cdot y' = x(\ln(\ln(x)))' + \ln(\ln(x))(x)'$$

Prod. Rule

$$\frac{1}{y}\cdot y' = x\cdot\left(\frac{1}{\ln(x)}\right)\left(\frac{1}{x}\right) + \ln(\ln(x))(1)$$

Double Chain Rule for first term

Isolate y' $\quad y' = \left[\left(\frac{1}{\ln(x)}\right) + \ln(\ln(x))\right]\cdot y$

Plug $y = [\ln(x)]^x$ $\quad \boxed{y' = [\ln(x)]^x\left[\frac{1}{\ln(x)} + \ln(\ln(x))\right]}$

b) Normal:

$f(x) = x^2 + 1$

$f'(x) = 2x$

$\boxed{y' = 2x}$

Notice: When logarithmic differentiation is unnecessary i.e. one can solve for y before differentiating, a factor of y will easily cancel at the end.

Logarithmic/Implicit Differentiation

$f(x) = x^2 + 1$

$y = x^2 + 1$

$\ln(y) = \ln(x^2 + 1)$,

$[\ln(y)]' = [\ln(x^2 + 1)]'$

$\frac{1}{y} \cdot y' = \frac{1}{(x^2+1)} \cdot (2x)$

$y' = \frac{1}{(x^2+1)} \cdot (2x)(y)$

$y' = \frac{1}{\cancel{(x^2+1)}} \cdot (2x)\cancel{(x^2+1)}$

$\boxed{y' = 2x}$

c) $f(x) = 3^x$

Normal:

$y = a^x$, $a = 3$

$y' = a^x \ln(a)$

$\boxed{y' = 3^x \ln(3)}$

Logarithmic:

$y = 3^x$

$\ln(y) = \ln(3^x)$

$\ln(y) = x\ln(3)$

$[\ln(y)]' = [x]'\ln(3)$

$\frac{1}{y} \cdot y' = (1)\ln(3)$

$y' = y\ln(3)$

$\boxed{y' = 3^x \ln(3)}$