Geometry Problems

1. Let A, B, C, D be points on a circle, in that order. Let AC and BD intersect in the point X. Show that $|AX| \cdot |XC| = |BX| \cdot |XD|$.

2. Let $\triangle ABC$ be a triangle. Show that the perpendicular bisectors of AB, BC and AC meet at a single point.

3. Let $\triangle ABC$ be a triangle. Show that the angle bisectors of $\angle ABC, \angle BAC$ and $\angle ACB$ meet at a single point.

4. A right circular cone (i.e, the normal shape of a cone) has a base of radius 1 and a height of radius 3. A cube is inscribed in the cone so that one face of the cube is contained inthe base of the cone. What is the side length of the cube?

5. Let $\triangle ABC$ be a triangle with circumcenter O, whose circumradius is R. Let D be the foot of the altitude dropped down from A. Let L be the second intersection of the line AO with the circumcircle (the first intersection is obviously A).

   (a) Show that triangles $\triangle ABD$ and $\triangle ALC$ are similar, and hence that $|AD| \cdot 2R = |AC| \cdot |AB|$.

   (b) Show that the area of $\triangle ABC$ is $\frac{|AB||AC||BC|}{4R}$.

6. Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$: the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta \to 0} |EF|$.

7. A line from vertex A of an equilateral triangle $\triangle ABC$ meets the opposite side BC in a point P and the circumcircle in Q. Prove that

$$\frac{1}{|PQ|} = \frac{1}{|BQ|} + \frac{1}{|CQ|}$$

8. A rectangle, $HOMF$, has side $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitutes, O the center of the circumscribed cirlce, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC?