Inequality Problems

1. Show that
$$1 \cdot 3 \cdot 5 \cdots (2n-1) \leq n^n$$

2. Show that if a, b, c are all positive, then
$$(a+b)(b+c)(a+c) \geq 8abc$$

3. Show that if $a_1, a_2, \ldots, a_n$ are real numbers such that $a_1 + a_2 + \cdots + a_n = 1$, then
$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{1}{n}$$

4. Show that if a, b, c are all positive, then
$$\sqrt{3(a+b+c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$$

5. Show that
$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{999999}{1000000} < \frac{1}{1000}$$
Hint: Square each side and "give a little" to create a telescoping product.

6. (1998 Putnam) Find the minimum value of
$$\frac{(x+1/x)^6 - (x^6 + 1/x^6) - 2}{(x+1/x)^3 + (x^3 + 1/x^3)}$$
for $x > 0$.

7. Show that for any integer n,
$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$$
Note: You may recognize these expressions: they approach e as $n \to \infty$.

8. If a, b and c are sides of a triangle, show that
$$\frac{a}{b+c-a} + \frac{b}{a+c-b} + \frac{c}{a+b-c} \geq 3$$

9. (2003 Putnam) Show that if $a_1, a_2, \ldots, a_n$ are non-negative real numbers, then
$$(a_1 a_2 \ldots a_n)^{1/n} + (b_1 b_2 \ldots b_n)^{1/n} \leq [(a_1 + b_1)(a_2 + b_2) \ldots (a_n + b_n)]^{1/n}$$

10. (2004 Putnam) Let m and n be positive integers. Show that
$$\frac{(m+n)!}{(m+n)^{m+n}} \leq \frac{m! \cdot n!}{m^m n^n}$$
Hint: The fastest way to do this is far too clever for its own good and uses the binomial formula. However, there are many different methods!