Modular Arithmetic Questions

1. Compute $5^{15} \pmod 7$ and $7^{13} \pmod{11}$.
2. Find the last two digits of 7^{100}.
3. Prove that if $x^3 + y^3 = z^3$ has a solution in integers, then one of the numbers must be a multiple of 7.
4. Show that an integer is divisible by 9 if and only if the sum of its digits is divisible by 9.
5. Prove that $n^7 - n$ is divisible by 42 for any n.
6. Given a set of 7 integers, show that there is a pair either whose sum or whose difference is divisible by 10.
7. Given n integers, show that there's a subset whose sum is divisible by n.
8. Prove that if p is a prime, then $(p-1)! \equiv -1 \pmod p$
9. Let f be a nonconstant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$.
10. The number $d_1 d_2 \dots d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1 e_2 \dots e_9$ is such that each of the nine 9-digit numbers formed by replacing just one of the digits d_i is $d_1 d_2 \dots d_9$ by the corresponding digit e_i $(1 \leq i \leq 9)$ is divisible by 7. The number $f_1 f_2 \dots f_9$ is related to $e_1 e_2 \dots e_9$ is the same way: that is, each of the nine numbers formed by replacing one of the e_i by the corresponding f_i is divisible by 7. Show that, for each i, $d_i - f_i$ is divisible by 7. [For example, if $d_1 d_2 \dots d_9 = 199501996$, then e_6 may be 2 or 9, since 199502996 and 199509996 are multiples of 7.]