Useful Polynomial Facts

1. If $P(x)$ is a polynomial of degree n in one variable, with complex coefficients, then it factors into linear terms:

   $$P(x) = C(x - r_1)(x - r_2)\cdots(x - r_n)$$

   where $r_1, \ldots, r_n$ are the roots of $P(x)$ – these are often complex numbers.

2. If $P(x)$ is a polynomial in one variable, and $P(a) = 0$, then $P(x)$ is divisible by $x - a$: that is, there exists $Q(x)$ such that

   $$P(x) = (x - a)Q(x)$$

3. A polynomial of degree n in one variable has at most n different roots.

4. If $P(x)$ is a polynomial of degree n in one variable with real coefficients, then $P(x)$ factors as a product of linear and quadratic terms with real coefficients.

5. If $P(x)$ and $D(x)$ are polynomials in one variable, then we can write

   $$P(x) = Q(x)D(x) + R(x)$$

   where the degree of $R(x)$ is less than the degree of $D(x)$. Here, $R(x)$ stands for the remainder of $P(x)$ when divided by $D(x)$.

6. In case anyone doesn't remember, here's the quadratic formula: the roots of $ax^2 + bx + c = 0$ are

   $$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

7. The set of polynomials of degree n is a vector space with basis $\{1, x, \ldots, x^n\}$; as this makes clear, it has dimension $n + 1$.

8. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, with roots $r_1, r_2, \ldots, r_n$. That is,

   $$P(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$$

   Then, we have that

   $$r_1 + r_2 + \cdots + r_n = (-1)a_{n-1}$$
   $$\vdots$$
   $$r_1 r_2 \ldots r_n = (-1)^n a_0$$

   (Figure out the remaining relationships by expanding out $(x - r_1)(x - r_2)\cdots(x - r_n)$ and comparing coefficients – they are a bit cumbersome to write down!)