# Polynomials Questions

1. Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers $a$. (Note: $\lfloor \nu \rfloor$ is the greatest integer less than or equal to $\nu$.)

2. Let $p(x)$ be a polynomial with integer coefficients. Assume that $p(a) = p(b) = p(c) = -1$, where $a, b, c$ are three different integers. Prove that $p(x)$ has no integral zeros.

3. Let $n$ be an even positive integer, and let $p(x)$ be an $n$-degree polynomial such that $p(-k) = p(k)$ for $k = 1, 2, ..., n$. Prove that there is a polynomial $q(x)$ such that $p(x) = q(x^2)$.

4. Let $k$ be a fixed positive integer. The $n$-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

5. Find polynomials $f(x)$,$g(x)$, and $h(x)$, if they exist, such that for all $x$,
$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

6. Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that
$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$
holds identically?

7. For each integer $m$, consider the polynomial
$$P_m(x) = x^4 - (2m + 4)x^2 + (m - 2)^2.$$
For what values of $m$ is $P_m(x)$ the product of two non-constant polynomials with integer coefficients?

8. Suppose that $a, b, c, A, B, C$ are real numbers, $a \neq 0$ and $A \neq 0$, such that
$$|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$$
for all real numbers $x$. Show that
$$|b^2 - 4ac| \leq |B^2 - 4AC|.$$

9. Let $n$ be a positive integer. Find the number of pairs $P, Q$ of polynomials with real coefficients such that
$$(P(X))^2 + (Q(X))^2 = X^{2n} + 1$$
and $\deg P > \deg Q$.

10. Let $p(z)$ be a polynomial of degree $n$, all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.