Separable Differential Equations

If we have a differential equation in $y = f(x)$ that can be rearranged as:

$$\frac{dy}{dx} = g(y)h(x)$$

then we have the following (illegal looking but correct!) way to solve it:

$$\begin{aligned} \frac{1}{g(y)}\ dy &= h(x)\ dx \\ \Rightarrow \int \frac{1}{g(y)}\ dy &= \int h(x)\ dx \end{aligned}$$

If you can actually do the indefinite integral and solve for y, we have a solution to the differential equation. Don't forget the arbitrary constant when integrating!

Example: Solve $f'(x) = f(x)^2 x^3$.
Solution: Rewriting this with y instead of $f(x)$ and $\frac{dy}{dx}$ instead of $f'(x)$, we get

$$\begin{aligned} \frac{dy}{dx} &= y^2 x^3 \\ \Rightarrow \frac{1}{y^2} dy &= x^3 dx \\ \Rightarrow \int \frac{1}{y^2} dy &= \int x^3 dx + C \\ \Rightarrow -\frac{1}{y} &= \frac{x^4}{4} + C \end{aligned}$$

Thus, solving, we get that

$$y = -\frac{1}{x^4/4 + C}$$

where C can be anything.