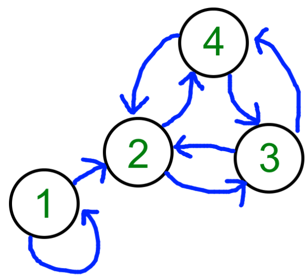
Homework 3

1. What is the transition matrix of the following random walk, if the probability of each arrow is 1/2?



2. Consider the following random walk on pairs (a, b) where each of a and b is either 0 or 1: that is, on $\{(0,0),(0,1),(1,0),(1,1)\}$. The rule is as follows: when we're at (a, b), we stay in place with probability $(a+b)/4$. The remaining probability is split equally between going to $(1-a, b)$ and $(a, 1-b)$.

   For example, if we're at $(1, 1)$, we stay at $(1, 1)$ with probability $1/2$, move to $(0, 1)$ with probability $1/4$ and move to $(1, 0)$ with probability $1/4$.

   (a) Write down an ordering for the states of this random walk and find the transition matrix under this ordering.

   (b) Let P be the transition matrix from part (a). Use Matlab to calculate a high power of this matrix. Would you guess this random walk has a limiting distribution? If so, write down what you think it is up to 3 decimal places.

3. Consider a random walk on the states $a_1, a_2, \ldots a_n$ as in class. Assume that we know that

$$P^2(a_i, a_j) = \mathbb{P}(a_i \xrightarrow{\text{2 steps}} a_j) \text{ and } P^3(a_i, a_j) = \mathbb{P}(a_i \xrightarrow{\text{3 steps}} a_j)$$

   for all i and j. (That is, we know the identity we proved in class for $m = 2$ and 3.) Use only these facts to show that

$$P^5(a_i, a_j) = \mathbb{P}(a_i \xrightarrow{\text{5 steps}} a_j)$$

4. Prove that $1^2 + 2^2 + \cdots + m^2 = \frac{m(m+1)(2m+1)}{6}$ for all positive integers m using induction.

5. [BONUS] What's wrong with the following induction argument?

   Claim: All horses are the same color.

   We use induction: we show that any m horses are the same color for any positive integer m.

   Base step: For $m = 1$, it's clear that one horse is the same color as itself.

   Inductive step: Assume any m horses are the same color, and let's show that any $m + 1$ horses are the same color. Line the $m + 1$ horses in a row. Then, the first m horses in this row are the same color (by assumption), and so are the last m horses. Since they overlap in the middle horses, all $m + 1$ horses must be the same color. Thus, we're done.