Homework 5

1. Consider two distributions ν_1, ν_2 on the finite state space Ω. In this question, we will prove the identity

$$\|\nu_1 - \nu_2\|_{TV} = \max_{A \subseteq \Omega} |\nu_1(A) - \nu_2(A)|$$

   (a) [5 pts] Define S to be the set of $x \in \Omega$ such that $\nu_1(x) \leq \nu_2(x)$. Show that

$$\begin{aligned} \|\nu_1 - \nu_2\|_{TV} &= \frac{1}{2}\left(\sum_{x \in S}(\nu_2(x) - \nu_1(x)) + \sum_{x \in S^c}(\nu_1(x) - \nu_2(x))\right) \\ &= \frac{1}{2}(\nu_2(S) - \nu_1(S) + \nu_1(S^c) - \nu_2(S^c)) \end{aligned}$$

   (b) [5 pts] Show that $\nu_2(S) - \nu_1(S) = \nu_1(S^c) - \nu_2(S^c)$, and combine this with part (a) to show that

$$\|\nu_1 - \nu_2\|_{TV} = \nu_2(S) - \nu_1(S) = |\nu_1(S) - \nu_2(S)|$$

   (c) [5 pts] Use part (b) to conclude that

$$\|\nu_1 - \nu_2\|_{TV} \leq \max_{A \subseteq \Omega} |\nu_1(A) - \nu_2(A)|$$

   (d) [5 pts] Now, show that for any set $A \subseteq \Omega$,

$$\nu_1(A \cap S) - \nu_2(A \cap S) \leq \nu_1(A) - \nu_2(A) \leq \nu_1(A \cap S^c) - \nu_2(A \cap S^c)$$

   (e) [5 pts] Use part (d) to show that for any $A \subseteq \Omega$, $|\nu_1(A) - \nu_2(A)| \leq \nu_2(S) - \nu_1(S)$ and hence by part (b)

$$|\nu_1(A) - \nu_2(A)| \leq \|\nu_1 - \nu_2\|_{TV}$$

   Combining this with part (c) shows the desired result!

   Notation: S^c is the *complement* of S – that is, all the elements of Ω that are not in S. Formally,

$$S^c = \{\text{all } a \in \Omega \text{ such that } a \notin S\}$$

   $X \cap Y$ is the *intersection* of sets X and Y – that is, all the elements that appear both in X and in Y. Formally,

$$X \cap Y = \{\text{all } a \text{ such that } a \in X \text{ and } a \in Y\}$$

2. For this question, you will be using the Matlab eigs function – for documentation, see http://www.mathworks.com/help/matlab/ref/eigs.html.

   Recall that to find the left eigenvectors of a matrix, it suffices to find the right eigenvectors of the transpose of A.

   The number of times an eigenvalue appears in the matrix D calculated by Matlab is the *multiplicity* of the eigenvalue – for us, the important fact is that if the multiplicity is 1, then the eigenvalue has a unique eigenvector up to scalar multiplication. When the multiplicity is higher, Matlab will provide two distinct eigenvectors for the eigenvalue.

   (a) [5 pts] Consider a Markov chain with transition matrix P

   $$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.25 & 0.75 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

   Use Matlab to find two distinct distributions ν satisfying $\nu P = \nu$ (that is, stationary distributions of P). Why is such a distribution not unique?

   Clarification: We have shown that under certain conditions, the distribution is unique. Please explain why these conditions do not hold here.

   (b) [5 pts] Now consider a Markov chain with transition matrix

   $$Q = \begin{bmatrix} 0 & 0.25 & 0 & 0.75 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.75 & 0 & 0.25 & 0 \end{bmatrix}$$

   Use Matlab to find whether this matrix has a unique stationary distribution π, and calculate what it is.

   (c) [5 pts] Use Matlab to check whether π is the limiting distribution for the above Markov chain. If it is not, explain how that is possible.

3. Consider a Markov chain with finite state space Ω and transition matrix P. For this question, let $\mathcal{T}(x)$ be the set of m such that $P^m(x, x) > 0$ – that is, the possible return times at x.

   (a) [5 pts] We say that a set S is *closed under addition* if for any $s_1, s_2 \in S$ (not necessarily distinct), $s_1 + s_2 \in S$. Show that $\mathcal{T}(x)$ is closed under addition.

   (b) [BONUS] It happens to be true that any set of positive integers that's closed under addition and whose gcd is 1 contains all integers greater than some number m. Use this to show that if the chain is irreducible and aperiodic, there exists an r such that $P^r(x, y) > 0$ for all $x, y \in \Omega$.