Homework 7

1. Consider the gambler's ruin walk. As in class, let f_k be the expected number of steps until the walk hits either 0 or n if it starts at k. Recall that these satisfy

$$f_k = \frac{1}{2}f_{k-1} + \frac{1}{2}f_{k+1} + 1, f_0 = f_n = 0$$

   In this question, we will show that $f_k = k(n-k)$, as stated in class.

   (a) [5 pts] Use induction to show that $f_k = kf_1 - k(k-1)$.

   (b) [5 pts] Use the fact that $f_n = 0$ to solve for f_1 and then use part (a) to find the formula for f_k.

2. Consider the gambler's ruin walk for a hesitant gambler: if the walk is at $k \neq 0$ or n, he flips a coin which lands heads with probability p. If the coin lands heads, the walk moves up with probability 1/2 and down with probability 1/2; if the coin lands tails, the walk stays in place. If the walk is at 0 or n, it stays there. (The question on the quiz was the case $p = 1/2$).

   (a) [5 pts] Let q_k be the probability that the walk ends at 0 if it starts at k. Find the equations that the q_k satisfy without simplifying.

   (b) [5 pts] Let g_k be the expected amount of time it takes the walk to end at either 0 or n if it starts at k. Find the equations that the g_k satisfy without simplifying.

3. (a) [2 pts] Let X be a random variable such that $\mathbb{P}(X = 1) = 1/2, \mathbb{P}(X = 2) = 1/3$ and $\mathbb{P}(X = 3) = 1/6$. Calculate $\mathbb{E}(X)$.

   (b) [3 pts] Let X be a random variable such that $\mathbb{P}(X = i) = 2/3^i$ for all positive integers i. Calculate $\mathbb{E}(X)$.

   (c) [5 pts] Prove that if X is a nonnegative integer valued random variable, $\mathbb{E}(X) = \sum_{i=1}^{\infty} \mathbb{P}(X \geq i)$.

4. Consider the pair of random variables (X, Y) that's distributed as:

$$(X,Y) = \begin{cases} (1,2) & \text{with probability } 1/2 \\ (1,3) & \text{with probability } 1/3 \\ (2,2) & \text{with probability } 1/12 \\ (1,3) & \text{with probability } 1/12 \end{cases}$$

   This is a coupling of some pair of distributions μ and ν.

   (a) [3 pts] What are the distributions μ and ν? (State both the state spaces and the probabilities of each state.)

(b) [5 pts] Write down another coupling of these distributions. (There are many – just pick one!)

5. Let P be the transition matrix for the gambler's ruin random walk on $\{1, 2, \ldots, 20\}$

   (a) [5 pts] Use a for loop to enter in P into Matlab. Print your code and output for grading.

   (b) [2 pts] Calculate P^{2000}.

   (c) [5 pts] Explain where the answer in (b) comes from.