Homework 9

1. Consider the following three stopping times for the random transposition walk. To illustrate how the stopping times work, we will be using the following example of a path the walk could take, starting at time 0: (here, L stands for the lefthand card and R stands for the righthand card.)

$$1234 \xrightarrow{L=1,R=2} 2134 \xrightarrow{L=3,R=2} 3124 \xrightarrow{L=4,R=3} 4123$$
$$\xrightarrow{L=1,R=4} 4321 \xrightarrow{L=3,R=3} 4321$$

   (i) **Stopping time** τ_1: Start with no marked cards. Every time a card appears in a transposition, mark it. Stop when all the cards are marked.

   In the above example, the set of marked cards would evolve as: $\emptyset \to \{1,2\} \to \{1,2,3\} \to \{1,2,3,4\} \to \{1,2,3,4\}$. Thus, $\tau_1 = 3$.

   (ii) **Stopping time** τ_2**:** Start with only card 1 marked. Every time a card is swapped with an already marked card, mark it. Stop when all the cards are marked.

   In the above example, the set of marked cards would evolve as: $\{1\} \to \{1,2\} \to \{1,2,3\} \to \{1,2,3,4\} \to \{1,2,3,4\}$. Thus, $\tau_2 = 3$.

   (iii) **Stopping time** τ_3**:** Start with only card 1 marked. Every time an unmarked righthand card is swapped with a marked lefthand card, mark it. Also mark a card every time it's swapped with itself.

   In the above example, the set of marked cards would evolve as: $\{1\} \to \{1,2\} \to \{1,2\} \to \{1,2,4\} \to \{1,2,3,4\}$. Thus, $\tau_3 = 4$.

   Now that you've read the descriptions of the stopping times, do the following questions:

   (a) [5 pts] One of the above stopping times is a strong stationary time. Check which one it is as you did in class – make sure to show your work for all of them!

   (b) [5 pts] Use induction to show that the marked cards are equally likely to appear in any particular order at a given time t.

   For example, if we know that at time t the marked cards are $\{1,3,4\}$, we would be showing that we're equally likely to see them in either of the orders

$$134, 143, 314, 341, 413, 431$$

   (c) [3 pts] Use part (b) to conclude that we indeed have a strong stationary time.

   (d) [BONUS – 10 pts] Calculate the expected value of the above strong stationary time and use it to find a bound on the mixing time of the random transposition walk.

2. In this question, we will learn how to input the transition matrix for the random transposition walk. With some modifications, these steps should enable you to also enter in matrices for other walks – for example, the random walks you are studying in your projects!

   We will need to use a list of permutations in order to construct the transition matrix. The idea is the following: we pick an ordering $(x_1, x_2, \ldots, x_{n!})$ for the permutations, then as usual we create a transition matrix to correspond to that ordering: that is, the (i, j) entry of our matrix will be the probability of going from x_i to x_j in one step. Luckily, the Matlab perms(v) function outputs a matrix whose rows are all the permutations of the vector v. We will use the ordering implicit in that function.

   (a) [2 pts] As you've probably seen, Matlab uses the notation $[1 : n]$ for the vector $[1, 2, 3, \ldots, n]$. Use this and the Matlab perms function as above to set L to the matrix whose rows are all the permutations of $\{1, 2, 3, 4\}$.

   (b) [2 pts] The Matlab 'ismember' command will find whether an element is a member of a list (feel free to read the documentation, as usual.) In particular, the command $[\sim, i]$ = ismember(r, A, 'rows') will set i to be the index of first row where the vector r appears in the matrix A. Use this to find which row the permutation $[1, 2, 3, 4]$ corresponds to in the matrix L above.

   (c) [5 pts] The Matlab randperm(n) command will output a random permutation of the numbers $\{1, 2, \ldots, n\}$. Set v to a random permutation of $\{1, 2, 3, 4\}$ and output v. Now, output a list of all the permutations you can get from v by picking a number with one hand, picking a number with the other hand, and swapping them. (This list should include 4 copies of the permutation $[1, 2, 3, 4]$ corresponding to picking the same number with both hands.)

   Hint: You will probably want to use two nested 'for' loops, as you'll want an index i for the first card and an index j for the second card.

   (d) [2 pts] We will now begin making the transition matrix for the walk. Initialize a 24 by 24 matrix P of all zeros by using the zeros function.

   (e) [10 pts] Make a for loop for $4! = 24$ turns. When we're at i, use the command $L(i, :)$ to pick out the ith row r_i of L. Then, as in part (c), construct every permutation which you can get by swapping two elements of r_i: at each such permutation, find its index j in L as in part (b). Finally, increase $P(i, j)$ by $1/16$.

3. (a) [5 pts] Using the matrix entered in above, find the absolute spectral gap for the random transposition walk for $n = 3, 4, 5$ and 6. You will want to use the eigs function.

   (b) [5 pts] Using the pattern from part (a), conjecture what the value of the absolute spectral gap is for general n. What upper and lower boundsxt will this give for the mixing time of the random transposition walk?