Combinatorics Questions

1. How many positive integers n are there such that n is an exact divisor of at last one of the numbers $10^{40}, 20^{30}$?

2. Let $\mathbf{S} = \{(a, b) | a = 1, 2, \ldots, n, b = 1, 2, 3\}$. A *rook tour* of $\mathbf{S}$ is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that

   (i) $p_i \in \mathbf{S}$,

   (ii) p_i and p_{i+1} are a unit distance apart, for $1 \leq i < 3n$,

   (iii) for each $p \in \mathbf{S}$ there is a unique i such that $p_i = p$. How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$?

3. Two distinct squares of the 8 by 8 chessboard are said to be adjacent if they have a vertex or side in common. Also, g is called a *possible gap* if for every numbering of the squares of the chessboard with all the integers $1, 2, \ldots, 64$, there exist two adjacent squares whose numbers differ by at least g. Determine the largest possible gap g.

4. A *transversal* of an $n \times n$ matrix A consists of n entries of A, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices A satisfying the following two conditions:

   (a) Each entry $\alpha_{i,j}$ of A is in the set $\{-1, 0, 1\}$.

   (b) The sum of the n entries of a transversal is the same for all transversals of A.

   An example of such a matrix A is

   $$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

   Determine with proof a formula for $f(n)$ of the form

   $$f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4,$$

   where the a_i's and b_i's are rational numbers.

5. A Dyck n-path is a lattice path of n upsteps $(1, 1)$ and n downsteps $(1, -1)$ that starts at the origin O and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis.

   Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n-1)$-paths.

6. Given a finite string S of symbols X and O, we write $\Delta(S)$ for the number of X's in S minus the number of O's. For example, $\Delta(XOOXOOX) = -1$. We call a string S **balanced** if every substring T of (consecutive symbols of) S has $-2 \leq \Delta(T) \leq 2$. Thus, $XOOXOOX$ is not balanced, since it contains the substring $OOXOO$. Find, with proof, the number of balanced strings of length n.

7. Let $A(n)$ denote the number of sums of positive integers

$$a_1 + a_2 + \cdots + a_r$$

which add up to n with

$$a_1 > a_2 + a_3, a_2 > a_3 + a_4, \ldots,$$
$$a_{r-2} > a_{r-1} + a_r, a_{r-1} > a_r.$$

Let $B(n)$ denote the number of $b_1 + b_2 + \cdots + b_s$ which add up to n, with

(a) $b_1 \geq b_2 \geq \cdots \geq b_s$,

(b) each b_i is in the sequence $1, 2, 4, \ldots, g_j, \ldots$ defined by $g_1 = 1$, $g_2 = 2$, and $g_j = g_{j-1} + g_{j-2} + 1$, and

(c) if $b_1 = g_k$ then every element in $\{1, 2, 4, \ldots, g_k\}$ appears at least once as a b_i.

Prove that $A(n) = B(n)$ for each $n \geq 1$.

(For example, $A(7) = 5$ because the relevant sums are $7, 6+1, 5+2, 4+3, 4+2+1$, and $B(7) = 5$ because the relevant sums are $4+2+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1$.)