Number Theory – All About Digits!

1. Find the last digit of $7^{7^{7^7}}$.

2. Let A be the sum of the digits of the number 4444^{4444} and B the sum of the digits of A. Compute the sum of the digits of B.

3. How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

4. What is the units (i.e., rightmost) digit of

$$\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor ?$$

   Here $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

5. The sequence of digits

$$123456789101112131415161718192021\ldots$$

   is obtained by writing the positive integers in order. If the 10^n-th digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

6. Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

7. Find the fifth digit from the end of $5^{5^{5^{5^5}}}$.

8. Let $N(k)$ be the number of integers n, $0 \leq n \leq 10^k$ digits can be permuted in such a way that they yield an integer divisible by 11. Prove that $N(2m) = 10N(2m-1)$ for all positive integers m.

9. For each positive integer n, let $a_n = 0$ (or 1) if the number of 1's in the binary representation of n is even (or odd), respectively. Show that there do not exist positive integers k and m such that

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j},$$

   for $0 \leq j \leq m-1$.