Sets and Operations

1. Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

2. Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

3. **(You can hand this one in)** Let S be a non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S the set $\{a^n : n = 1, 2, 3, \ldots\}$ is finite. Must S be a group?

4. A is a subset of a finite group G and A contains more than one half of the elements of G. Prove that each element of G is the product of two elements of A. T

5. Prove or disprove the following statement: if F is a finite set with two or more elements, then there exists a binary operation $*$ on F such that for all $x, y, z \in F$,

   (a) $x * z = y * z$ implies $x = y$.

   (b) $x * (y * z) \neq (x * y) * z$.

6. Let S be the set of ordered triples (a, b, c) of distinct elements of a finite set A. Suppose that

   (a) $(a, b, c) \in S$ if and only if $(b, c, a) \in S$;

   (b) $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$;

   (c) (a, b, c) and (c, d, a) are both in S if and only if (b, c, d) and (d, a, b) are both in S.

   Prove that there exists a one-to-one function g from A to $\mathbb{R}$ such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$.

7. Let G be a group with identity e and $\phi : G \to G$ a function such that

   $$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

   whenever $g_1 g_2 g_3 = e = h_1 h_2 h_3$. Prove that there exists an element $a \in G$ such that $\psi(x) = a\phi(x)$ is a homomorphism (i.e. $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$).

8. Let $\mathcal{M}$ be a set of real $n \times n$ matrices such that

   (i) $I \in \mathcal{M}$, where I is the $n \times n$ identity matrix;

   (ii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB \in \mathcal{M}$ or $-AB \in \mathcal{M}$, but not both;

   (iii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB = BA$ or $AB = -BA$;

   (iv) if $A \in \mathcal{M}$ and $A \neq I$, there is at least one $B \in \mathcal{M}$ such that $AB = -BA$.

   Prove that $\mathcal{M}$ contains at most n^2 matrices.