A rapid review of complex function theory

T. PERUTZ

I have followed W. Rudin, *Real and complex analysis* (chapter 10) and H. Priestley, *Introduction to complex analysis*.

Notational conventions

- $\mathbb{C}$ is the complex plane.
- Open discs: $D(a;r) := \{z \in \mathbb{C} : |z-a| < r\}$.
- Closed discs: $\bar{D}(a;r) = \{z \in \mathbb{C} : |z-a| \leq r\}$.
- Punctured open discs: $D^*(a;r) = \{z \in \mathbb{C} : 0 < |z-a| < r\}$.

1 Holomorphic functions

Definition 1.1 Let G be an open set in $\mathbb{C}$. A function $f\colon G \to \mathbb{C}$ is called *holomorphic* if, at every point $z \in G$, the complex derivative

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

exists as a complex number.

Write $\mathcal{O}(G)$ for the set of holomorphic functions on G.

1.1 Basic properties of holomorphic functions

- If G and H are open sets in $\mathbb{C}$, and $H \subset G$, then the restriction to H of a holomorphic function on G is again holomorphic; thus one has a restriction map $\mathcal{O}(G) \to \mathcal{O}(H)$.
- The sum, product, chain and quotient rules hold for holomorphic functions, just as for differentiable functions of a real variable.
- Since sums and products of holomorphic functions are again holomorphic, $\mathcal{O}(G)$ is a commutative ring. Since constant functions are holomorphic, there is a ring homomorphism $\mathbb{C} \to \mathcal{O}(G)$, making $\mathcal{O}(G)$ a commutative, associative $\mathbb{C}$-algebra.

1.2 The derivative as a linear map

If f is holomorphic, one has

$$f(z+h) = f(z) + f'(z)h + \varepsilon_z(h), \qquad \varepsilon_z(h) \in o(h), \tag{1}$$

i.e., $\varepsilon_z(h)/h \to 0$ as $h \to 0$. Conversely, if $f(z+h) = f(z) + g(z)h + \varepsilon_z(h)$, where $\varepsilon_z(h) \in o(h)$ for all z, then f is holomorphic with derivative g.

From (1) one sees that a holomorphic function is differentiable in the real sense, as a function $G \to \mathbb{C} = \mathbb{R}^2$—that is, there is an $\mathbb{R}$-linear map (the derivative) $D_z f\colon \mathbb{C} \to \mathbb{C}$ such that $f'(z+h) =$

$f(z) + (D_z f)(h) + \varepsilon_z(h)$ where $\varepsilon_z \in o(h)$. The derivative is $(D_z f)(h) = f'(z)h$. It is not only $\mathbb{R}$-linear *but also* $\mathbb{C}$*-linear*.

Complex linearity of the derivative is equivalent to the *Cauchy–Riemann equation*

$$D_z(ih) = iD_z(h) \tag{2}$$

In terms of real and imaginary parts, a $\mathbb{C}$-linear map $\mathbb{C} \to \mathbb{C}$ is represented by a real matrix of form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

On the other hand, writing $z = x + yi$, $h = h_1 + h_2 i$, and $f = u + vi$ (all variables on the right real) one has

$$(D_f z)\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

so we see that the Cauchy–Riemann equation can be written in the familiar form

$$u_x = v_y, \quad u_y = -v_x.$$

Conversely, a function $f \colon G \to \mathbb{C}$ that is $\mathbb{R}$-differentiable and satisfies the Cauchy–Riemann equation is holomorphic, since the derivative $D_z f$ must be multiplication by a complex number $f'(z)$.

While differentiability of a function $f \colon G \to \mathbb{R}^2$ is a regularity (or smoothness) condition, that is not true of holomorphy. For f to be holomorphic it must also satisfy a PDE, and that leads to properties of holomorphic functions that are drastically more special than those of real differentiable functions.

1.3 Power series

For a complex power series $\sum_{n=0}^{\infty} a_n (z-c)^n$, centered at c, we define the *radius of convergence*

$$R = \sup\left\{ |z-c| : \sum_{n=0}^{\infty} a_n (z-c)^n \text{ converges} \right\} \in [0, \infty].$$

Theorem 1.2 *Let $\sum_{n=0}^{\infty} a_n (z-c)^n$ be a power series with radius of convergence R. Then*

1. $R^{-1} = \limsup |a_n|^{1/n}$.
2. *The sum $\sum_{n=0}^{\infty} a_n (z-c)^n$ converges for $z \in D(c;R)$, to a function $f(z)$. The convergence is absolute.*
3. *The convergence to f is uniform on compact subsets of $D(c;R)$.*
4. *The series $\sum_{n=1}^{\infty} n a_n (z-c)^{n-1}$ also has radius of convergence R, and so converges on $D(c;R)$ to a function g.*
5. $f \in \mathcal{O}(D(c;R))$, *and* $f' = g$.

The proof is basically a comparison with geometric series.

Proof (1 and 2) It follows from the root test that if $\limsup |a_n|^{1/n} |z-c| > 1$ then the series diverges, while if $\limsup |a_n|^{1/n} |z-c| < 1$ then it converges absolutely (hence converges).

(3) It suffices to prove uniform convergence on $D(c;\rho)$ where $\rho < R$. On this disc one has $|a_n(z-c)^n| \leq M_n$, where $M_n = |a_n|\rho^n$. But $\sum M_n < \infty$, and so uniform convergence follows from the Weierstrass M-test.

(4) Apply (1), noting that $\limsup\{(n+1)^{1/n}|a_{n+1}|^{1/n}\} = \limsup |a_n|^{1/n}$.

(5) We may assume $c = 0$. Absolute convergence of f and g validates the formula

$$\frac{f(z)-f(w)}{z-w} - g(w) = \sum_{n\geq 1} a_n \left(\frac{z^n - w^n}{z-w} - nw^{n-1} \right).$$

for distinct $z, w \in D(c;R)$. One has

$$\frac{z^n - w^n}{z-w} - nw^{n-1} = (z-w)\frac{\partial}{\partial w}\left(\frac{z^n - w^n}{z-w} - nw^{n-1} \right) = (z-w)(z^{n-2} + 2z^{n-3}w + \cdots + (n-1)w^{n-1})$$

(the sum on the right is understood to be zero if $n = 1$). Thus if $|z|$ and $|w|$ are less than $\rho < R$, one has

$$\left| \frac{z^n - w^n}{z-w} - nw^{n-1} \right| \leq \frac{1}{2} n(n-1)\rho^{n-2}|z-w|$$

and

$$\left| \frac{f(z)-f(w)}{z-w} - g(w) \right| \leq \left(\sum_{n=2}^{\infty} \frac{1}{2} n(n-1)\rho^{n-2} \right) |z-w| = \frac{1}{(1-\rho)^3}|z-w|,$$

which goes to 0 as $z \to w$. □

2 Cauchy's theorem

2.1 Path integrals

Definition 2.1 If $G \subset \mathbb{C}$ is an open set, a *smooth path* in G is a compact non-empty interval $[a,b]$ and a C^1 map $\gamma\colon [a,b] \to G$. A *path* in G is a compact non-empty interval $[a,b]$ and a continuous map $\gamma[a,b] \to G$ such that $[a,b]$ is the union of finitely many closed subintervals J such that $\gamma|_J$ is a smooth path. A *loop* is a path such that $\gamma(a) = \gamma(b)$.

For a smooth path $\gamma\colon [a,b] \to G$, and for a continuous function $f\colon G \to \mathbb{C}$, we define the integral of f along γ to be

$$\int_\gamma f(z)\,dz := \int_a^b f \circ \gamma(t) \cdot \gamma'(t)\,dt.$$

We also write $\int_\gamma f$ for $\int_\gamma f(z)\,dz$.

This integral behaves in the expected fashion under reparametrization, and is additive under joining paths. For a general path γ, we define $\int_\gamma f(z)\,dz$ by summing integrals over subintervals on which γ is smooth.

The fundamental theorem of calculus holds: if $F \in \mathcal{O}(G)$ then $\int_\gamma F'(z)dz = F(\gamma(b)) - F(\gamma(a))$. In particular:

Proposition 2.2 *If $\gamma\colon [a,b] \to G$ is a loop and $F \in \mathcal{O}(G)$ then $\int_\gamma F'(z)dz = 0$.*

2.2 Proving Cauchy's theorem

Given three points v_1, v_2 and v_3 in $\mathbb{C}$, we define the triangle $T = T(v_1, v_2, v_3)$ to be their convex hull. Its *boundary* ∂T is the loop formed as the join of the three directed intervals $[v_1, v_2]$, $[v_2, v_3]$ and $[v_3, v_1]$, each parametrized with unit speed.

Lemma 2.3 (Cauchy's theorem for a triangle) *Let $f \in \mathcal{O}(G)$; let v_1, v_2 and v_3 be points of G whose convex hull T is contained in G. Then $\int_{\partial T} f = 0$.*

Proof Let $v_{12} = \frac{1}{2}(v_1 + v_2)$; similarly define v_{23} and v_{31}. We subdivide the triangle T into four congruent triangles:

$$T^1 = T(v_1, v_{12}, v_{31}), \quad T^2 = T(v_2, v_{23}, v_{12}), \quad T^3 = T(v_3, v_{31}, v_{23}), \quad T^4 = T(v_{12}, v_{23}, v_{31}).$$

Then $\int_{\partial T} = \sum_{k=1}^4 \int_{\partial T^k}$, because of cancelations. Let T_1 be a triangle T^k which maximizes $\left|\int_{\partial T^k} f\right|$. Thus

$$\left|\int_{\partial T} f\right| \leq 4 \left|\int_{\partial T_1} f\right|.$$

Similarly subdivide T_1, and iterate the process to obtain a nested sequence of triangles T_N such that

$$\left|\int_{\partial T} f\right| \leq 4^N \left|\int_{\partial T_N} f\right|.$$

Let z^* be the unique point in $\bigcap T_N$. One has $f(z) = f(z^*) + f'(z^*)(z - z^*) + \varepsilon(z)$, where $\varepsilon(z)/|z - z^*| \to 0$ as $z \to z^*$. Now

$$\int_{\partial T_N} \big(f(z^*) + f'(z^*)(z - z^*)\big)\, dz = 0$$

by Proposition 2.2, and so

$$\int_{\partial T_N} f = \int_{\partial T_N} \varepsilon.$$

Thus

$$\left|\int_{\partial T} f\right| \leq 4^N \left|\int_{\partial T_N} \varepsilon\right| \leq 4^N \cdot 2^{-N} \mathsf{length}\,(\gamma) \sup_{z \in \partial T_N} |\varepsilon(z)|.$$

For any $\delta > 0$, we can find a disc centered on z^* on which $|\varepsilon(z)| \leq |z - z^*|\delta$. Thus if N is sufficiently large one has $\sup_{z \in \partial T_N} |\varepsilon(z)| \leq 2^{-N}\delta$, whence

$$\left|\int_{\partial T} f\right| \leq \mathsf{length}\,(\gamma) \cdot 2^N \cdot \sup_{z \in \partial T_N} |\varepsilon(z)| \leq \mathsf{length}(\gamma) \cdot \delta.$$

□

Theorem 2.4 (Cauchy's theorem for a star-domain) *Let G be a star-domain, and $f \in \mathcal{O}(G)$. Then for any closed path γ in G one has $\int_\gamma f = 0$.*

Proof Let z^* be a star-point; thus for all $z \in G$, the interval $[z^*, z]$ lies in G. Define $F(z) = \int_{[z,z^*]} f(z)\, dz$. It suffices to show that $F' = f$.

Take $z \in G$, and let $\delta > 0$ be small enough that $D(z; \delta) \subset G$. By Cauchy's theorem for a triangle one has, when $|h| < \delta$,

$$F(z + h) - F(z) = \int_{[z,z+h]} f.$$

Applying this formula to the expression $f(z+h) = f(z) + f'(z)h + \varepsilon_z(h)$, one obtains

$$F(z+h) = F(z) + f(z)h + \frac{1}{2}f'(z)h^2 + \int_0^1 h\varepsilon_z(th)\,dt,$$

from which one obtains $F' = f$. □

Corollary 2.5 (deformation theorem) *Let G be an open set, and let γ_0 and γ_1 be closed paths that are homotopic in G (through closed paths). Then $\int_{\gamma_0} f = \int_{\gamma_1} f$ for all $f \in \mathcal{O}(G)$.*

One takes the homotopy to be piecewise C^1 in an appropriate sense. The proof is to subdivide the homotopy into many small homotopies whose images lie inside discs in G, and then to apply Cauchy's theorem for a star-domain to each of those.

Corollary 2.6 (Cauchy's theorem for simply connected regions) *Let G be a simply connected open set in $\mathbb{C}$, and let γ be a closed path in G. Then $\int_\gamma f = 0$ for all $f \in \mathcal{O}(G)$.*

Corollary 2.7 *Let G be a simply connected open set in $\mathbb{C}$. Then every holomorphic function on G possesses an antiderivative on G.*

One constructs the antiderivative F as $F(z) = \int_{\gamma_z} f$, where γ_z is an arbitrary path connected a basepoint $z_0 \in G$ to z.

2.3 From Green's theorem to Cauchy's theorem.

Recall that Green's theorem states that if D is a compact set in the plane whose boundary ∂D is a smooth curve then, for the counterclockwise orientation of ∂D, one has

$$\iint_D (G_x - F_y)dx\,dy = \int_{\partial D} F\,dx + G\,dy$$

provided that F and G are C^1 functions. If $f = u + vi$ is holomorphic and C^1, one gets

$$\mathrm{Re} \int_{\partial D} f(z)dz = \int_{\partial D} u\,dx + v\,dy = \iint_D (v_x - u_y)\,dx\,dy.$$

But $v_x - u_y = 0$ by the Cauchy–Riemann equations, so $\mathrm{Re} \int_{\partial D} f(z)dz = 0$. Similarly $\mathrm{Im} \int_{\partial D} f(z)dz = 0$. This shows that $\int_{\partial D} f(z)dz = 0$, a form of Cauchy's theorem. (This argument is essentially the one that Cauchy himself used.)

The drawback of this approach is that it requires one to assume f is C^1. It is true that every holomorphic function is C^1, but the usual proof is via Cauchy's theorem. To avoid circularity, if one takes this approach one must *either* assume that one's functions are holomorphic and C^1 (which is aesthetically displeasing, though not really problematic in practice), *or* find an alternative proof that holomorphic functions are C^1. There is such an alternative, known as *Weyl's lemma* (or as *elliptic regularity*), and we will prove it (for other reasons) in this course.

3 Power series representations of holomorphic functions

3.1 Integral formulae

Theorem 3.1 (Cauchy's integral formula) *Let G be an open set containing the disc $D(z;R)$ and take $f \in \mathcal{O}(G)$. Then, for $r \in (0,R)$, one has*

$$f(z) = \frac{1}{2\pi i}\int_{C(z;r)} \frac{f(w)}{w-z}dw.$$

Here $C(z;r)$ denotes the circular path $[0,2\pi] \ni t \mapsto z + re^{it}$.

Proof One has

$$\frac{1}{2\pi i}\int_{C(z;r)} \frac{f(w)}{w-z}dw - f(z) = \frac{1}{2\pi i}\int_{C(z;r)} \frac{f(w)-f(z)}{w-z}dw.$$

Since the integrand in the latter expression is holomorphic on the punctured disc $D^*(z;r)$, one has by the deformation theorem that the integral does not depend on $r \in (0,R)$. But we can find a disc centered at z on which

$$\left|\frac{f(w)-f(z)}{w-z} - f'(z)\right| \leq |w-z|.$$

Taking r smaller than the radius of this disc, we find that $\left|\frac{1}{2\pi i}\int_{C(z;r)}\frac{f(w)}{w-z}dw - f(z)\right| \leq |f'(z)|r + r^2$, and the right-hand side goes to zero as $r \to 0$. □

Theorem 3.2 (holomorphic functions are analytic) *Let $f \in \mathcal{O}(D(c;R)$. Then one has $f(z) = \sum_{n\geq 0} a_n(z-c)^n$ on $D(c;R)$, where for any $r \in (0,R)$ one has*

$$a_n = \frac{1}{2\pi i}\int_{C(c;r)} \frac{f(z)}{(z-c)^{n+1}}dz$$

Proof We use Cauchy's integral formula and the deformation theorem. Take $\delta > 0$ small, and then take $r \in (|z-c|, R)$. Then

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{C(z;\delta)} \frac{f(w)}{w-z}dw \quad \text{(for small } \delta > 0) \\ &= \frac{1}{2\pi i}\int_{C(c;r)} \frac{f(w)}{w-z}dw \\ &= \frac{1}{2\pi i}\int_{C(c;r)} \frac{f(w)}{w-c}\left(\frac{1}{1-\frac{z-c}{w-c}}\right)dw \\ &= \frac{1}{2\pi i}\int_{C(c;r)} \frac{f(w)}{w-c}\sum_{n\geq 0}\left(\frac{z-c}{w-c}\right)^n dw \\ &= \sum_{n\geq 0}\left(\frac{1}{2\pi i}\int_{C(c;r)} \frac{f(w)}{(w-c)^{n+1}}dw\right)(z-c)^n. \end{aligned}$$

In the formula for a_n, we know by the deformation theorem that any $r \in (0,R)$ will give the same integral. □

Corollary 3.3 (Cauchy's formula for derivatives) *Holomorphic functions are C^∞. For $f \in \mathcal{O}(G)$ one has*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{C(z;r)} \frac{f(w)}{(w-z)^{n+1}}dw$$

for small $r > 0$.

3.2 Discrete zeros

Theorem 3.4 *Let $f \in \mathcal{O}(G)$, where $G \subset \mathbb{C}$ is an open set. If f is not the zero-function then $f^{-1}(0)$ is discrete in G.*

Proof Suppose $f(c) = 0$, and let $f(z) = \sum_{n=1}^{\infty} a_n(z-c)^n$ be the Taylor series representation of f near c. If f is not the zero-function then there is some non-vanishing coefficient a_n. Let m be the smallest integer such that a_m is non-vanishing. Then we have $f(z) = (z-a)^m g(z)$, where $g(z) = \sum_{n=m}^{\infty} a_n(z-c)^{n-m}$. The function g is holomorphic near a, since it is a power series with positive radius of convergence, equal to that for $\sum_{n=1}^{\infty} a_n(z-c)^n$, and $g(a) = a_m \neq 0$. Thus g is non-vanishing near a, and hence f is non-vanishing in a punctured neighborhood of a. □

The integer m that appears in the proof is called the *multiplicity* of the zero, and is denoted by $\mathsf{mult}(f;c)$.

3.3 Liouville's theorem

Theorem 3.5 (Liouville's theorem) *If $f \in \mathcal{O}(\mathbb{C})$ is bounded then f is constant.*

Proof Say $|f| \leq M$. We have

$$f'(z) = \frac{1}{\pi i}\int_{C(0;r)} \frac{f(w)}{(w-z)^2}dw,$$

for any $r > |z|$. Taking $r > 2|z|$, we have $|w-z| \geq |w| - |z| \geq r/2$ for $w \in C(0;r)$. Thus $|f'(z)| \leq \frac{1}{\pi} \cdot 2\pi r \cdot 4Mr^{-2} = 8Mr^{-1}$, whence $f'(z) = 0$. Since z is arbitrary, f is constant. □

4 Morera's theorem and its consequences

Theorem 4.1 (Morera's theorem) *If f is a continuous function on the open set $G \subset \mathbb{C}$, and $\int_{\partial T} f = 0$ for all triangles $T \subset G$, then $f \in \mathcal{O}(G)$.*

Proof On a disc $D(c;r) \subset G$, we can define an antiderivative F for f by the formula $F(z) = \int_{[c,z]} f$ (the proof that $F' = f$ is as in the proof of Cauchy's theorem for a triangle). Since F is holomorphic, so is its derivative f. □

4.1 Removable singularities

Corollary 4.2 *Let f be holomorphic on an open set G containing a punctured disc $D^*(a;r)$ on which $|f|$ is bounded. Then f extends to a holomorphic function on $G \cup \{a\}$.*

In this situation, we say that f has a *removable singularity* at a.

Proof Let $g(z) = (z-a)f(z)$. Then $g \in \mathcal{O}(G)$, and $g(z) \to 0$ as $z \to a$. Hence g can be continuously extended to $G \cup \{a\}$ by setting $g(a) = 0$. One now checks that Morera's theorem applies to g on $G \cup \{a\}$: take a triangle $T \subset G$. If $a \notin T$ then $\int_{\partial T} g = 0$ by Cauchy's theorem. If $a \in \partial T$ then we can replace T by a closed curve γ not containing a at the cost of an arbitrarily small change in the integral; thus $\int_{\partial T} g = 0$. If $a \in \operatorname{int}(T)$ then by the deformation theorem we can replace T by an arbitrarily small triangle T' whose interior contains T. But $\int_{\partial T'} g$ can be made arbitrarily small, hence $\int_{\partial T} g = 0$.
By Morera, g is holomorphic over a. But $g(a) = 0$, from which it follows that $(z-a)^{-1}g(z)$ is again holomorphic over a. This is the sought extension of f. □

4.2 Limits of sequences of holomorphic functions

Theorem 4.3 *Let $(f_n)_{n=0}^{\infty}$ be a sequence of holomorphic functions on an open set G converging, uniformly on compact subsets of G, to a limit f. Then $f \in \mathcal{O}(G)$, and $f_n'(z) \to f'(z)$.*

Proof We may work in an open set U contained in a compact subset of G, so that convergence is uniform. Uniform convergence implies both that the limit f is continuous, and that $\int_\gamma f_n \to \int_\gamma f$ for paths γ in U. Taking γ to be an arbitrary triangle, we have $\int_\gamma f_n = 0$ by Cauchy, so $\int_\gamma f = 0$, so f is holomorphic by Morera. That $f_n'(z) \to f'(z)$ follows from Cauchy's integral formula for derivatives. □

In practice, one often wants to combine this theorem with the Weierstrass M-test, which states that if one has a sequence of functions (f_n) on U, and if there are constants $M_n \geq 0$ such that $|f_n| \leq M_n$ on U, and $\sum M_n < \infty$, then the sum $f(z) = \sum_{n=0}^{\infty} f_n(z)$ converges uniformly on U:

Corollary 4.4 *Let $(f_n)_{n=0}^{\infty}$ be a sequence of holomorphic functions on an open set U. Assume that there are constants $M_n \geq 0$ such that $|f_n| \leq M_n$ on U, and $\sum M_n < \infty$. Then the sum $\sum f_n$ converges to a holomorphic function f, and $f' = \sum f_n'$.*

5 Isolated singularities

Theorem 5.1 *Consider the annulus $A = A(c; r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z-c| < r_2\}$, and take $f \in \mathcal{O}(A)$. Then one can write*

$$f(z) = \sum_{n<0} a_n (z-c)^n + \sum_{n \geq 0} a_n (z-c)^n,$$

where $a_n = \frac{1}{2\pi i} \int_{C(c;r)} \frac{f(z)}{(z-c)^{n+1}} dz$.

Sketch of proof Fix $z \in A$. Choose r_1' and r_2' so that $r_1 < r_1' < |z-c| < r_2' < r_2$. Then one has

$$f(z) = \frac{1}{2\pi i}\int_{C(c;r_1')} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{C(c;r_2')} \frac{f(w)}{w-z}dw,$$

as one sees by splitting the closed annulus bounded by $C(c;r_1')$ and $C(c;r_2')$ into two contractible regions, the first containing z, the second not, and applying Cauchy's integral formula and Cauchy's theorem to these respective regions. One then develops binomial expansions for the two integrands, like in the proof that holomorphic functions are analytic. □

If $f \in \mathcal{O}(G)$, where G contains a punctured disc centered at c, the *residue* of f at c, denoted $\mathsf{res}\{f;c\}$, is the coefficient a_{-1} in the Laurent expansion. Thus

$$\mathsf{res}\{f;c\} = \frac{1}{2\pi i}\int_{C(c;r)} f(z)\,dz$$

for any small $r > 0$.

Theorem 5.2 (Residue theorem) *Let D be a compact, simply connected region bounded by a loop ∂D. Let $a_1, \ldots, a_k$ be interior points of D, let $D^* = \operatorname{int} D \setminus \{a_1, \ldots, a_k\}$, and take $f \in \mathcal{O}(D^*)$. Then*

$$\frac{1}{2\pi i}\int_{\partial D} f = \sum_j \mathsf{res}\{f;a_j\}.$$

Sketch proof The argument is to consider a null-homotopy $\{\gamma_t\}$ of the loop $\gamma_0 = \partial D$. As t increase towards 1, the a_j are expelled from the region bounded by γ_t. Using the deformation theorem, one finds that $\int_{\partial D} f$ is the sum of the integrals of f around small loops encircling the a_j, i.e., $2\pi i$ times the sum of the residues. □

5.1 Counting zeros

Theorem 5.3 (Argument principle) *Let D be a compact, simply connected region bounded by a loop ∂D. Assume that f is holomorphic on a neighborhood of D, and non-zero on ∂D. Then*

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)}dz = \sum_{z \in D\colon f(z)=0} \mathsf{mult}\,(f;z). \tag{3}$$

$$\frac{1}{2\pi i}\int_{\partial D} z \cdot \frac{f'(z)}{f(z)}dz = \sum_{z \in D\colon f(z)=0} f(z) \cdot \mathsf{mult}\,(f;z). \tag{4}$$

Proof The logarithmic derivative f'/f is holomorphic except at the discrete set $f^{-1}(0)$. Enumerate the finite set $f^{-1}(0) \cap D$ as $(a_1, \ldots, a_k)$. By the residue theorem, $\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)}dz$ is the sum of the residues of f'/f at the points a_j. Thus, to prove the first formula, it suffices to show that if f has a zero of multiplicity m at a point a then $\mathsf{res}\,\{f'/f, a\} = m$. We may write $f(z) = (z-a)^m g(z)$, where g is holomorphic near a and $g(a) \neq 0$. Then

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)},$$

which makes it clear that the residue is m. The proof of the second formula works in the same way. □

Corollary 5.4 (Open mapping theorem) *Let G be a connected open set and $f \in \mathcal{O}(G)$ non-constant. Then f is an open mapping, i.e., it maps open sets to open sets.*