Two lectures about the Seiberg–Witten equations on symplectic 4-manifolds

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1 Vortices, monopoles and Seiberg–Witten invariants

As a general introduction to Seiberg–Witten theory, I suggest [Mor]. For a rapid but insightful survey, I recommend [Don], which I have drawn on in these lectures. I will not cover the 3-dimensional aspects of Seiberg–Witten theory—the story of monopole Floer homology—nor the relation of the theory to contact 1-forms and their periodic Reeb orbits. For these aspects, I recommend Hutchings's article [Hut].

1.1 Applications of the SW equations to symplectic topology

A special feature of the 4-dimensional Seiberg–Witten equations, one which the closely-related instanton equations do not possess, is that they behave in a special way in the presence of a symplectic form. This feature has been systematically exploited in the work of Taubes, which plays a role in many striking results by a number of authors.

Let (X,ω) be a closed symplectic 4-manifold, and K_X its canonical class. The proofs of the following results depend on the fact that **the Seiberg–Witten equations have a canonical solution on X—the unique solution in the canonical** Spin^c**-structure**.

- (Taubes [T94].) X is not diffeomorphic to the connected sum of two oriented 4-manifolds with $b^+ > 0$.
- (Kronheimer–Mrowka [KM94]; Morgan–Szabó–Taubes [MST]; Ozsváth–Szabó [OS].) Any embedded symplectic surface in X minimizes genus within its homology class.
- (Kronheimer [Kro].) For each integer $k \geq 0$, there is a symplectic 4-manifold (X_k,ω_k) such that the homotopy group $\pi_{2k+1}(\mathsf{diff}(X_k)/\mathsf{aut}(X_k,\omega_k))$ is non-vanishing.
- (Morgan–Szabó [MS]; Bauer [Bau]; T.-J. Li [Li].) K_X torsion implies $b^+ \leq 3$. If also $\pi_1(X) = \{1\}$ then X has the homotopy-type of a K3 surface.

Gromov [Gro] taught us how to use pseudo-holomorphic curves to probe symplectic topology. The sharpest results are for those symplectic 4-manifolds which are contain pseudo-holomorphic curves through every point. The power of this technique is drastically increased when it is combined with Seiberg–Witten theory.

The next collection of results hinge on existence theorems for pseudo-holomorphic curves which derive from Taubes's theorem "$\mathsf{SW} = \mathsf{Gr}$", which says that the Seiberg–Witten invariants of the 4-manifold X are equal to certain Gromov–Witten invariants, and thereby implies **existence** results for pseudo-holomorphic curves.

- (Taubes [T96].) K_X is Poincaré dual to an embedded symplectic surface.
- (Taubes [T96].) Up to isomorphism and scale, $\mathbb{C}P^2$ admits a unique symplectic structure.
- (Taubes [T96].) If X contains a smoothly embedded sphere S of self-intersection -1, then it contains a homologous sphere which is symplectically embedded, hence can be blown down.
- (A.-K. Liu [Liu].) A minimal symplectic 4-manifold with $K \cdot \omega < 0$ is either $\mathbb{C}P^2$ or a symplectic S^2-bundle over S^2. One with $K \cdot K < 0$ is a symplectic S^2-bundle over a surface of genus ≥ 2. In each case, the symplectic form is determined up to symplectomorphism by its cohomology class.
- (Biran [Bir].) Certain symplectic 4-manifolds X admit full packings by symplectic balls, i.e., given $k \gg 0$, there exist, for any $\epsilon > 0$, k disjoint embeddings of symplectic balls of equal radius, filling all but ϵ of the volume of X.

1.2 The vortex equations

We'll approach the 4-dimensional Seiberg–Witten equations via their 2-dimensional reduction: the *vortex equations* [JT, Gar, Wit].

Background: Let (Σ, g) be a closed, connected Riemannian surface. The metric g gives rise to a conformal (or complex) structure j and to an area form $\alpha = \mathrm{vol}_g$. Let $L \to \Sigma$ be a hermitian line bundle of degree $d = c_1(L)[\Sigma]$.

Fields: Pairs (A, ϕ), where A is a $U(1)$-connection in L, and ϕ is a C^∞ section of L.

Parameter: $\tau \in \mathbb{R}$.

Equations: The *τ-vortex equations* read

$$\overline{\partial}_A \phi = 0 \qquad \text{in } \Omega^{0,1}(\Sigma; L); \tag{1}$$

$$iF_A = (\tau - |\phi|^2)\alpha \qquad \text{in } \Omega^2(\Sigma). \tag{2}$$

We shall call τ the *Taubes parameter*. Solutions are called *τ-vortices*; they form a space $\widetilde{\mathsf{vor}}_\tau(L)$.

Lemma 1.1 *Let $T = \int \tau\alpha/2\pi$.*

- *If $T < d$ then no τ-vortex exists.*
- *If $T = d$ then $(A,\phi) \in \widetilde{\mathsf{vor}}_\tau(L)$ iff $\phi = 0$ and $F_A = \tau\omega$.*
- *If $T > d$ and $(A,\phi) \in \widetilde{\mathsf{vor}}_\tau(L)$ then ϕ is not identically zero.*

Proof Chern–Weil tells us that $\int iF_A/2\pi = d$. The equations then give $T - \frac{1}{2\pi}\int |\phi|^2\alpha = d$, so $T < d$ with equality iff $\phi = 0$. □

The gauge group $\mathcal{G} = C^\infty(\Sigma, U(1))$ operates on pairs (A,ϕ) by

$$u \cdot (A,\phi) = (A - u^{-1}du, u\phi),$$

preserving the τ-vortices. Let $\mathsf{vor}_\tau(L)$ be the quotient $\widetilde{\mathsf{vor}}_\tau(L)/\mathcal{G}$. If, as we shall now assume, $T > d$, then the action on τ-vortices is free. Given a τ-vortex (A,ϕ), the vortex equations linearize to

$$\bar{\partial}_A\,\psi + a^{0,1}\phi = 0, \quad ida + 2\,\mathrm{Re}\langle\phi,\psi\rangle = 0 \quad \text{for } \; a \in i\Omega^1(\Sigma),\; \psi \in \Omega^0(L).$$

We can impose in addition the gauge condition

$$d^*a - 2i\,\mathrm{Im}\langle\phi,\psi\rangle = 0$$

which gives a local slice through the $\mathcal{G}$-action. The linearized vortex equations plus the gauge-fixing equationdefine a linear operator $\mathcal{L}$ which is elliptic, hence Fredholm, and which one can show to be *surjective*. Up to zeroth order terms, it is the direct sum of operators $\bar{\partial}_A \colon \Omega^0(L) \to \Omega^1(L)$, which has index $2d + \chi(\Sigma)$ (by Riemann–Roch), and $d^* \oplus d \colon \Omega^1 \to \Omega^0 \oplus \Omega^2$, which has index $-\chi(\Sigma)$ (b Hodge theory). Hence $\dim\ker\mathcal{L} = 2d$. So:

Proposition 1.2 *$\mathsf{vor}_\tau(L)$ is naturally a $2d$-dimensional manifold.*

The operator $\bar{\partial}_A$ makes L a *holomorphic* line bundle (the holomorphic local sections are those in the kernel of $\bar{\partial}_A$). The first equation says that L is a holomorphic line bundle and ϕ a holomorphic section. Hence, when $T > d$, one has a map

$$\mathsf{vor}_\tau(L) \to \left\{ (\mathcal{L},\phi) \,\middle|\, \begin{array}{c} \mathcal{L} \text{ a holomorphic structure on } L \\ \phi \not\equiv 0 \text{ a holomorphic section} \end{array} \right\} / C^\infty(\Sigma, \mathbb{C}^*).$$

The complex moduli space on the right is better known as the symmetric product $\mathrm{Sym}^d(\Sigma) = \Sigma^{\times d}/S_d$, and the map is

$$v \colon \mathsf{vor}_\tau(L) \to \mathrm{Sym}^d(\Sigma), \quad [A,\phi] \mapsto \phi^{-1}(0).$$

Theorem 1.3 *[JT, Gar] The map v is a diffeomorphism.*

Remark When $d = 0$, one has a unique vortex, up to gauge (corresponding to $\mathrm{Sym}^0\,\Sigma = \{\emptyset\}$). The connection A is flat of trivial holonomy (and so trivializes the bundle) and ϕ is constant.

In the 'Taubes limit' $\tau \gg 0$, vortices (A,ϕ) 'localize' along their zero-sets:

Theorem 1.4 *One has* $|F_A| + |d_A\phi| \leq c \exp\left(-\frac{\tau^{1/2}}{c}\, dist(\cdot, \phi^{-1}(0))\right)$, *where the constant* c *depends only on* (Σ, L).

Thus, the interesting behavior happens very close to the zero-set $\phi^{-1}(0)$. Elsewhere, A is essentially flat—and so by second vortex equation, $|\phi|^2 - \tau$ is close to zero—while ϕ is essentially covariant-constant.

In fact [T99], one can use this estimate to construct an inverse to v, for $\tau \gg 0$, by constructing approximate vortices and improving them to true vortices using the implicit function theorem.

The approximate solutions are given by pasting rescaled solutions on $\mathbb{C}$ into Σ near the given point $\mathbf{x} \in \mathrm{Sym}^d(\Sigma)$, extending them to Σ using cutoff functions so that, far from $\mathbf{x}$, A is flat and ϕ covariant-constant of norm-squared τ.

1.3 The Seiberg–Witten equations

Background:

We now work over a closed, oriented Riemannian 4-manifold X. We first fix a Spin^c-structure $\mathfrak{s}$. This is a choice from an $H^2(X;\mathbb{Z})$-torsor. We think of $\mathfrak{s}$ in differential-geometric terms as:

- A pair $\mathbb{S}^\pm \to X$ of hermitian 2-plane bundles, called the positive and negative spinor bundles.
- A bundle isomorphism $\rho\colon T^*X \otimes \mathbb{C} \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{S}^+, \mathbb{S}^-)$, called Clifford multiplication, satisfying the relation that makes $\mathbb{S}^+_x \oplus \mathbb{S}^-_x$ a module for the Clifford algebra $\mathrm{Cliff}(T^*_xX)$:
$$\rho(f)^\dagger\rho(e) + \rho(e)^\dagger\rho(f) = -2g(e,f)\mathrm{id}_{\mathbb{S}^+}.$$

One can then define ρ on complex 2-forms by
$$\rho(e \wedge f) = \frac{1}{2}\left(\rho(e)^\dagger\rho(f) - \rho(f)^\dagger\rho(e)\right) \in \mathrm{End}(\mathbb{S}^+).$$

One checks that $\rho(\Lambda^+) = \mathfrak{su}(\mathbb{S}^+)$ and $\rho(\Lambda^-) = 0$.

Fields: (A, ϕ). Here $\phi \in \Gamma(\mathbb{S}^+)$, i.e., ϕ is a positive spinor, while A is a Clifford connection in $\mathbb{S}^+$. A $U(2)$-connection is a Clifford connection if $d_A(\rho(\lambda)) = \rho(\nabla\lambda)$ when ∇ is the Levi-Civita connection in T^*X. A connection A in $\mathbb{S}^+$ induces a connection A^t in the line bundle $\Lambda^2\mathbb{S}^+$, and $A \mapsto A^t$ defines a bijection between Clifford connections and $U(1)$-connections in $\Lambda^2\mathbb{S}^+$.

Parameters: $\eta \in \Omega^2_X$ is a closed 2-form. A_0 is a reference Clifford connection.

The Seiberg–Witten equations $SW(\mathfrak{s}, \eta)$:

These read

$$D_A^+\phi = 0 \quad \text{in } \Gamma(\mathbb{S}^-); \tag{3}$$

$$\rho(F_{A^t} + i\eta)^+ = (\phi^* \otimes \phi)_0 \quad \text{in } i\,\mathfrak{su}(\mathbb{S}^+). \tag{4}$$

We add the Coulomb gauge-fixing equation

$$d^*(A^t - A_0^t) = 0 \quad \text{in } i\Omega^0(X). \tag{5}$$

We now explain the terms:

$D_A^+ = \sum_j \rho(e_j)\nabla_{A,e_j} \colon \Gamma(\mathbb{S}^+) \to \Gamma(\mathbb{S}^-)$ is a Dirac operator (here (e_1, e_2, e_3, e_4) is a local oriented orthonormal frame for T^*X).

The $U(1)$-connection A^t has curvature $F_{A^t} \in i\Omega^2(X)$. The second equation is in $i\mathfrak{su}(\mathbb{S}^+)$, the trace-free hermitian endomorphisms. The term $\phi^* \otimes \phi$ is a hermitian endomorphism of $\mathbb{S}^+$. The symbol $(\cdot)_0$ means the trace-free part. Thus, if $\phi = \alpha f_1 + \beta f_2$ in a local unitary frame (f_1, f_2) for $\mathbb{S}^+$, then

$$(\phi^* \otimes \phi)_0 = \begin{bmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha\bar{\beta} \\ \bar{\alpha}\beta & \frac{1}{2}(|\beta|^2 - |\alpha^2|) \end{bmatrix}.$$

1.4 The SW invariants

The linearized Seiberg–Witten equations with Coulomb gauge-fixing are *elliptic*, hence Fredholm. Discarding zeroth-order terms—which do not affect the symbol, nor the Fredholm index—the linearized equations read

$$(d^+ + d^*)(a) = 0, \quad D_A^+\psi = 0.$$

For generic metrics g, the moduli space $\mathcal{M}(\mathfrak{s}, \eta)$ of solutions modulo gauge is naturally a smooth manifold, except at *reducible* solutions $(A, \phi = 0)$. When $b^+ > 0$, generic metrics have the property that there are no reducible solutions. Then $\mathcal{M}(\mathfrak{s}, \eta)$ is globally smooth, and in fact orientable; one obtains an orientation from an orientation of the vector space $\mathcal{H}^+_g \oplus \mathcal{H}^1_g$ of g-harmonic self-dual 2-forms plus 1-forms on X. It is a truly remarkable feature of the SW equations that $\mathcal{M}(\mathfrak{s}, \eta)$ is also *compact*.

The Seiberg–Witten invariant $\mathsf{SW}_X(\mathfrak{s}) \in \mathbb{Z}$ is defined, essentially, as the fundamental class $[\mathcal{M}(\mathfrak{s}, \eta)]$ in $H_*(\mathcal{B}^*)$, where $\mathcal{B}^*$ is the space of gauge-equivalence classes of irreducible pairs (A, ϕ) (so $\phi \not\equiv 0$). Explicitly, fix $x \in X$, let $\mathcal{L}_x \to \mathcal{B}^*$ be the principal $U(1)$-bundle with total space $\mathcal{L}_x$ given by pairs (A, ϕ) modulo gauge transformations u with $u(x) = 1$. Let $c = c_1(\mathcal{L}_x) \in H^2(\mathcal{B}^*)$. When $\dim \mathcal{M}(\mathfrak{s}, \eta) = 2d$, we define

$$\mathsf{SW}_X(\mathfrak{s}) = \langle [\mathcal{M}(\mathfrak{s}, \eta)], c^d \rangle.$$

When $\dim \mathcal{M}(\mathfrak{s}, \eta)$ is odd, we define $\mathsf{SW}_X(\mathfrak{s}) = 0$.

When $b^+ > 1$, $\mathsf{SW}_X(\mathfrak{s})$ is independent both of the 2-form η and the metric g, and so defines an invariant of X. That is because an interpolating path (η_t, g_t) defines an oriented cobordism in $\mathcal{B}^*$ between the moduli spaces for (η_0, g_0) and (η_1, g_1).

When $b^+ = 1$, there is a subtlety, which is that in such interpolating paths one may encounter reducible solutions. The space of pairs (η, g) is divided into two ‘chambers’, and the SW-invariant depends on the chamber; there is a ‘wall-crossing formula’ which specifies the difference between the value of SW_X in the two chambers.

2 Second lecture: Seiberg–Witten invariants of symplectic 4-manifolds

2.1 The SW equations on symplectic 4-manifolds

We let (X,ω) be a symplectic 4-manifold, J a compatible almost complex structure, and $g=\omega(\cdot,J\cdot)$ the associated metric. For this metric, ω is self-dual, hence harmonic, and of type $(1,1)$ with respect to J.

There is a canonical Spin^c-structure $\mathfrak{s}_{\mathsf{can}}$ with spinor bundles

$$\mathbb{S}^+ = \mathbf{1}\oplus\Lambda^{0,2},\quad \mathbb{S}^- = \Lambda^{0,1}.$$

The operators

$$D^{\pm} = \sqrt{2}\left(\bar\partial\oplus\bar\partial^*\right) : \Gamma(\mathbb{S}^{\pm})\to\Gamma(\mathbb{S}^{\mp})$$

have the property that $D^+\circ D^-$ and $D^-\circ D^+$ are Laplacians (a property of the symbol). Moreover, D^- is the formal adjoint to D^+. As a consequence, the symbol ρ of D^+ defines a Clifford multiplication $\rho\colon T^*X\to\mathrm{Hom}(\mathbb{S}^+,\mathbb{S}^-)$ for which D^+ is a Dirac operator.

Any other Spin^c-structure takes the form $L\otimes\mathfrak{s}_{\mathsf{can}}$, where L is a hermitian line bundle. Thus

$$\mathbb{S}^+_L = L\oplus(\Lambda^{0,2}\otimes L),\quad \mathbb{S}^+ = \Lambda^{0,1}\otimes L.$$

For any $U(1)$-connection B in L, the operators

$$D^{\pm}_B = \frac{1}{\sqrt{2}}\left(\bar\partial_B\oplus\bar\partial^*_B\right) : \Gamma(\mathbb{S}^{\pm}_L)\to\Gamma(\mathbb{S}^{\mp}_L)$$

are the positive and negative Dirac operators for $L\otimes\mathfrak{s}_{\mathsf{can}}$; the symbol of D^+_B defines Clifford multiplication.

The SW equations. The Seiberg–Witten fields are now a triple (α,β,B), where $\alpha\in\Gamma(L)$, $\beta\in\Omega^{0,2}(L)$, and B is a $U(1)$-connection in L.

It proves useful to add a perturbation term $\tau\omega+iR$, with a 'Taubes parameter' $\tau>0$. Here R is the $(1,1)$-part of the curvature of the connection in $\Lambda^2\mathbb{S}$ induced by the Levi-Civita connection. The equations read

$$\bar\partial_B\,\alpha = -\,\bar\partial^*_B\,\beta \tag{6}$$

$$F^{02}_B = \bar\alpha\beta \tag{7}$$

$$iF^{11}_B\cdot\omega = |\beta|^2-|\alpha|^2+\tau. \tag{8}$$

2.2 Taubes's constraints

Theorem 2.1 *[T95] Let (X,ω) be a closed symplectic manifold. Consider the SW equations (6, 7, 8). If $c_1(L)\cdot\omega < 0$, or if $c_1(L)\cdot\omega = 0$ and L is a non-trivial line bundle, then there are no solutions when $\tau \gg 0$. When L is trivial and $\tau \gg 0$ there is a unique solution up to gauge.*

Proof [Don] We shall need the identity

$$\bar\partial_B\,\bar\partial_B f = F_B^{02} f + N_J(\partial_B f),$$

where $N_J\colon \Lambda^{0,1} \to \Lambda^{0,2}$ is the Nijenhuis tensor, which according to the Newlander-Nirenberg theorem measures the failure of integrability of J to a holomorphic structure. We also need a Weitzenböck formula,

$$\nabla_B^*\nabla_B = 2\,\bar\partial_B^*\,\bar\partial_B + iF_B^{11}\cdot\omega.$$

We now compute

$$\begin{aligned}
\int_X |\nabla_B\alpha|^2 d\mu &= 2\int_X \langle \bar\partial_B^*\,\bar\partial_B\,\alpha,\alpha\rangle d\mu + \int_X (iF_B^{11}\cdot\omega)|\alpha|^2 d\mu\\
&= -2\int_X \langle \bar\partial_B^*\,\bar\partial_B\,\beta,\alpha\rangle d\mu + \int_X (|\beta|^2-|\alpha|^2+\tau)|\alpha|^2 d\mu\\
&= -2\int_X \langle \beta, \bar\partial_B^*\,\bar\partial_B\,\alpha\rangle d\mu + \int_X (|\beta|^2-|\alpha|^2+\tau)|\alpha|^2 d\mu\\
&= -2\int_X \langle \beta, N_J(\partial_B\alpha)\rangle d\mu - 2\int_X \langle\beta, F_B^{02}\alpha\rangle d\mu + \int_X (|\beta|^2-|\alpha|^2+\tau)|\alpha|^2 d\mu\\
&= -2\int_X \langle \beta, N_J(\partial_B\alpha)\rangle d\mu - 2\int_X |\alpha|^2|\beta|^2 d\mu + \int_X (|\beta|^2-|\alpha|^2+\tau)|\alpha|^2 d\mu\\
&= -2\int_X \langle \beta, N_J(\partial_B\alpha)\rangle d\mu - \int_X |\alpha|^2|\beta|^2 d\mu + \int_X (-|\alpha|^2+\tau)|\alpha|^2 d\mu\\
&= -2\int_X \langle \beta, N_J(\partial_B\alpha)\rangle d\mu - \int_X |\alpha|^2|\beta|^2 d\mu - \int_X (\tau-|\alpha|^2)^2 d\mu + \tau\int_X (\tau-|\alpha|^2) d\mu.
\end{aligned}$$

Now, most of the terms on the right are manifestly non-positive. The Nijenhuis term is one exception. The other exception is $\int_X (\tau-|\alpha|^2)d\mu$. But, by the third SW equation and Chern–Weil,

$$\begin{aligned}
\int_X (\tau-|\alpha|^2)d\mu &= \int_X (iF_B^{11}\cdot\omega - |\beta|^2)d\mu\\
&= \int_X iF_B\wedge\omega - |\beta|^2 d\mu\\
&= 2\pi c_1(L)\cdot[\omega] - \int_X |\beta|^2 d\mu,
\end{aligned}$$

and so we obtain

$$\begin{aligned}
&\int_X |\nabla_B\alpha|^2 d\mu + \int_X (\tau + |\alpha|^2)|\beta|^2 d\mu \\
&\quad + \int_X (\tau - |\alpha|^2)^2 d\mu - 2\pi c_1(L)\cdot[\omega] = -2\int_X \langle \beta, N_J(\partial_B\alpha)\rangle d\mu \\
&\qquad\qquad \le C\left(\int_X |\beta|^2 d\mu\right)^{1/2}\left(\int_X |\nabla_B\alpha|^2 d\mu\right)^{1/2} \\
&\qquad\qquad \le \tfrac{1}{2}C^2\int_X |\beta|^2 d\mu + \tfrac{1}{2}\int_X |\nabla_B\alpha|^2 d\mu.
\end{aligned}$$

Rearranging,

$$\tfrac{1}{2}\int_X |\nabla_B\alpha|^2 d\mu + \int_X (\tau - \tfrac{1}{2}C^2 + |\alpha|^2)|\beta|^2 d\mu + \int_X (\tau - |\alpha|^2)^2 d\mu - 2\pi c_1(L)\cdot[\omega] \le 0.$$

Let us take $\tau > C^2/2$, and assume also that $-c_1(L)\cdot[\omega] \ge 0$. Then each term on the left is ≥ 0, and so must be zero. That is,

$$\beta = 0; \quad \tau - |\alpha|^2 = 0; \quad \nabla_B\alpha = 0.$$

Thus α is a nowhere-vanishing section of L, which is therefore a trivial line bundle, and the only solution in this line bundle is the canonical monopole. □

The canonical monopole will be cut out transversely when the metric is chosen generically. As a result, we have the following

Corollary 2.2

$$\mathsf{SW}_X(\mathfrak{s}_{\mathsf{can}}) = \pm 1$$

If $c_1(L)\cdot[\omega] > 0$, *or* $c_1(L)\cdot[\omega] = 0$ *with* L *non-trivial, we have*

$$\mathsf{SW}_X(L\otimes\mathfrak{s}_{\mathsf{can}}) = 0.$$

2.3 The Taubes limit

The 'Taubes limit' refers to the limit $\tau\to\infty$ in the equations with perturbation term $\tau\omega$. Taubes [T96] establishes that, in a sequence of solutions (α_n,β_n,B_n) to the $\tau_n\omega$-equations, $\tau_n\to\infty$, there is a subsequence which has the property that $\beta_n\to 0$, and moreover, that $\bar\partial_B^*\beta_n\to 0$. A glance at the equations then shows that $F_{B_n}^{2,0}\to 0$, and that $\bar\partial_{B_n}\alpha_n\to 0$.

If the almost complex structure J were integrable, the condition $F_B^{0,2}=0$ would say that $\bar\partial_B$ defines a holomorphic structure in L, and the equation $\bar\partial_B\alpha = 0$ would say that α is a holomorphic section. In our almost complex setting, one does not expect to find any solutions to $\bar{\partial}_B \alpha = 0$, but the sequence (B_n, α_n) give approximate solutions to this equation. In particular, the zero-set $\alpha_n^{-1}(0)$, which one can take to be an embedded surface in X, is approximately holomorphic.

Taubes shows, more precisely, that there is a subsequence such that $|\beta_n| \to 0$, and such that everything of interest takes place close to the surfaces $\alpha_n^{-1}(0)$. One has

$$|\beta_n|^2 + |F_{B_n}^{0,2}| + |\nabla_{B_n}\beta|^2 + (1 - |\alpha_n|^2)/\tau_n^2 \leq c \exp[-\tau_n \mathrm{dist}(\cdot, \alpha_n^{-1}(0))/c].$$

Using this and other estimates, he proves that $iF_B/2\pi$ converges as a current to a closed $(1,1)$-current $\mathcal{F}$.

In a particularly difficult step, he shows that $\mathcal{F}$ is the Dirac-delta current δ_C for a cycle $C = \sum n_i C_i$, where the C_i are disjoint, embedded J-holomorphic curves in X, and $n_i > 0$. The zero-sets $\alpha_n^{-1}(0)$ converge in the Gromov–Hausdorff topology to $\bigcup C_i$.

The proof of this result uses the positivity properties of $\mathcal{F}$, and applies a regularity result from geometric measure theory to recognize it as a holomorphic curve.

2.4 SW = Gr

On a symplectic 4-manifold, we can define a bijective map

$$h\colon \mathrm{Spin^c}(X) \to H_2(X;\mathbb{Z}), \quad L \otimes \mathfrak{s}_{\mathsf{can}} \mapsto PD(c_1(L)).$$

Theorem 2.3 (Taubes) *For any symplectic 4-manifold (X, ω), one has, for a canonical homology orientation, and in the chamber containing $\tau\omega$ for $\tau \gg 0$ when $b^+ = 1$,*

$$\mathsf{SW}_X(\mathfrak{s}) = \mathsf{Gr}\,(h(\mathfrak{s})).$$

Here $\mathsf{Gr}\,(A)$ is a Gromov–Witten-type invariant defined in [T96b]. It counts J-holomorphic curves, possibly reducible or disconnected, representing the class A, passing through d generic points when the Fredholm index of the moduli space is $2d$. The definition uses some special features of dimension 4: for a generic J, all J-holomorphic representatives of the homology class A have disjoint irreducible components, and moreover, each component is an embedded smooth curve unless it has genus 1 and self-intersection 0, or genus 0 and negative self-intersection. In the latter case, the image of the curve is smoothly embedded, but the parametrization is a multiple cover. The curve-count is integral (it does not use virtual methods), and is defined in relatively straightforward way except for the square-zero tori, for which the definition is quite subtle.

The first part of the proof [T96] is the 'Taubes limit', which we have already discussed: a sequence of solutions to the $\tau_n\omega$ SW equations gives rise to a J-holomorphic curve whose Dirac delta current is the limit of $iF_{B_n}/2\pi$. The second part [T99] shows that, given a holomorphic curve C of the kind counted by Gr, one can obtain it as the Taubes limit of Seiberg–Witten monopoles. This is done as follows.

Say $C = \sum n_i C_i$, where n_i is the multiplicity of the ith component. On each normal fibre N_x to C_i, one considers solutions to the τ-vortex equations, of degree n_i, which decay at infinity. These form a moduli space identified with $\mathrm{Sym}^{n_i}(N_x)$; as x varies, these moduli spaces form a bundle of complex manifolds over C_i. The almost complex structure J gives rise to an almost complex structure in the total space of this bundle, with holomorphic fibres. A holomorphic section of the bundle can be viewed as an approximate solution to the SW equations, defined in a tubular neighborhood of C. In particular, when $n_i = 1$, one can simply take the zero-section. One extends the approximate solution to all of X via a cutoff function, so that, far from C, one has a flat connection B_n, a covariant-constant section α_n of L, with $|\alpha_n| = \tau_n$, and $\beta_n = 0$. Using the implicit function theorem, one argues that this approximate monopole lies close to a true monopole.

The third part of the proof [T99b] is the numerical comparison of the invariants.

2.5 Duality

When $b^+ > 1$, the SW invariants are invariant under conjugation of Spin^c-structures:

$$\mathsf{SW}_X(\bar{\mathfrak{s}}) = \pm \mathsf{SW}_X(\mathfrak{s}).$$

(The sign can be made precise.) The reason is simple: the unperturbed SW equations are preserved by conjugation. One has $h(\bar{\mathfrak{s}}) = K_X - h(\mathfrak{s})$, where $K_X = -c_1(TX)$ is the canonical class. Passing to the Taubes limit on a symplectic manifold does not commute with conjugation, and the following corollary of $\mathsf{SW} = \mathsf{Gr}$ is highly non-trivial:

$$\mathsf{Gr}(A) = \pm \mathsf{Gr}(K_X - A).$$

In particular, $\mathsf{Gr}(0) = 1$ (we count the empty curve), and so $\mathsf{Gr}(K_X) = \pm 1$. Thus the canonical class K_X has a J-holomorphic representative.

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