A topological semigroup structure on the space of actions modulo weak equivalence.

Peter Burton

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Abstract

We introduce a topology on the space of actions modulo weak equivalence finer than the one previously studied in the literature. We show that the product of actions is a continuous operation with respect to this topology, so that the space of actions modulo weak equivalence becomes a topological semigroup.

1 Introduction.

Let Γ be a countable group and let (X, μ) be a standard probability space. All partitions considered in this note will be assumed to be measurable. If a is a measure-preserving action of Γ on (X, μ) and $\gamma \in \Gamma$ we write γ^a for the element of $\mathrm{Aut}(X, \mu)$ corresponding to γ under a. Let $\mathrm{A}(\Gamma, X, \mu)$ be the space of measure-preserving actions of Γ on (X, μ). We have the following basic definition, due to Kechris.

Definition 1. *For actions $a, b \in \mathrm{A}(\Gamma, X, \mu)$ we say that a is **weakly contained** in b if for every partition $(A_i)_{i=1}^n$ of (X, μ), finite set $F \subseteq \Gamma$ and $\epsilon > 0$ there is a partition $(B_i)_{i=1}^n$ of (X, μ) such that*

$$\left|\mu\left(\gamma^a A_i \cap A_j\right) - \mu\left(\gamma^b B_i \cap B_j\right)\right| < \epsilon$$

*for all $i, j \leq n$ and all $\gamma \in F$. We write $a \prec b$ to mean that a is weakly contained in b. We say a is **weakly equivalent** to b and write $a \sim b$ if we have both $a \prec b$ and $b \prec a$. $\sim$ is an equivalence relation and we write $[a]$ for the weak equivalence class of a.*

For more information on the space of actions and the relation of weak equivalence, we refer the reader to [3]. Let $\mathrm{A}_\sim(\Gamma, X, \mu) = \mathrm{A}(\Gamma, X, \mu)/\sim$ be the set of weak equivalence classes of actions. Freeness is invariant under weak equivalence, so the set $\mathrm{FR}_\sim(\Gamma, X, \mu)$ of weak equivalence classes of free actions is a subset of $\mathrm{A}_\sim(\Gamma, X, \mu)$.

Given $[a], [b] \in \mathrm{A}_\sim(\Gamma, X, \mu)$ with representatives a and b consider the action $a \times b$ on $\left(X^2, \mu^2\right)$. We can choose an isomorphism of $\left(X^2, \mu^2\right)$ with (X, μ) and thereby regard $a \times b$ as an action on (X, μ). The weak equivalence class of the resulting action on (X, μ) does not depend on our choice of isomorphism, nor on the choice of representatives. So we have a well-defined binary operation $\times$ on $\mathrm{A}_\sim(\Gamma, X, \mu)$. This is clearly associative and commutative. In Section 2 we introduce a new topology on $\mathrm{A}_\sim(\Gamma, X, \mu)$ which is finer than the one studied in [1], [2] and [4]. We call this the fine topology. The goal of this note is to prove the following result.

Theorem 1. *$\times$ is continuous with respect to the fine topology, so that in this topology $(\mathrm{A}_\sim(\Gamma, X, \mu), \times)$ is a commutative topological semigroup.*

In [?], Tucker-Drob shows that for any free action a we have $a \times s_\Gamma \sim a$, where s_Γ is the Bernoulli shift on $\left([0,1]^\Gamma, \lambda^\Gamma\right)$ with λ being Lebesgue measure. Thus if we restrict attention to the free actions there is additional algebraic structure.

Corollary 1. *With the fine topology,* $(\mathrm{FR}_\sim(\Gamma, X, \mu), \times)$ *is a commutative topological monoid.*

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2 Definition of the fine topology.

Fix an enumeration $\Gamma = (\gamma_s)_{s=1}^\infty$ of Γ. Given $a \in \mathrm{A}(\Gamma, X, \mu)$, $t, k \in \mathbb{N}$ and a partition $\mathcal{A} = (A_i)_{i=1}^k$ of X into k pieces let $M_{t,k}^{\mathcal{A}}(a)$ be the point in $[0,1]^{t \times k \times k}$ whose s, l, m coordinate is $\mu\left(\gamma_s^a A_l \cap A_m\right)$. Endow $[0,1]^{t \times k \times k}$ with the metric given by the sum of the distances between coordinates and let d_H be the corresponding Hausdorff metric on the space of compact subsets of $[0,1]^{t \times k \times k}$. Let $C_{t,k}(a)$ be the closure of the set

$$\left\{ M_{t,k}^{\mathcal{A}}(a) : \mathcal{A} \text{ is a partition of } X \text{ into } k \text{ pieces } \right\}.$$

We have $a \sim b$ if and only if $C_{t,k}(a) = C_{t,k}(b)$ for all t, k. Define a metric d_f on $\mathrm{A}_\sim(\Gamma, X, \mu)$ by

$$d_f([a],[b]) = \sum_{t=1}^\infty \frac{1}{2^t} \left(\sup_k d_H\left(C_{t,k}(a), C_{t,k}(b)\right) \right).$$

This is clearly finer than the topology on $\mathrm{A}_\sim(\Gamma, X, \mu)$ discussed in the references.

Definition 2. *The topology induced by d_f is called the the **fine topology**.*

We have $[a_n] \to [a]$ in the fine topology if and only if for every finite set $F \subseteq \Gamma$ and $\epsilon > 0$ there is N so that when $n \geq N$, for every $k \in \mathbb{N}$ and every partition $(A_l)_{l=1}^k$ of (X, μ) there is a partition $(B_l)_{l=1}^k$ so that

$$\sum_{l,m=1}^k \left|\mu\left(\gamma^{a_n} A_l \cap A_m\right) - \mu\left(\gamma^a B_l \cap B_m\right)\right| < \epsilon$$

for all $\gamma \in F$ and $l, m \leq k$.

3 Proof of the theorem.

We begin by showing a simple arithmetic lemma.

Lemma 1. *Suppose I and J are finite sets and $(a_i)_{i \in I}, (b_i)_{i \in I}, (c_j)_{j \in J}, (d_j)_{j \in J}$ are sequences of elements of $[0,1]$ with $\sum_{i \in I} a_i = 1$, $\sum_{j \in J} d_j = 1$, $\sum_{i \in I} |a_i - b_i| < \delta$ and $\sum_{j \in J} |c_j - d_j| < \delta$. Then $\sum_{(i,j) \in I \times J} |a_i c_j - b_i d_j| < 2\delta$.*

Proof. Fix i. We have

$$\begin{aligned}\sum_{j\in J}|a_ic_j-b_id_j| &\leq \sum_{j\in J}(|a_ic_j-a_id_j|+|d_ja_i-d_jb_i|)\\ &=\sum_{j\in J}(a_i|c_j-d_j|+d_j|a_i-b_i|)\\ &\leq \delta a_i+|a_i-b_i|.\end{aligned}$$

Therefore

$$\sum_{(i,j)\in I\times J}|a_ic_j-b_id_j|\leq\sum_{i\in I}(a_i\delta+|a_i-b_i|)\leq 2\delta.$$

□

We now give the main argument.

Proof of Theorem 1. Suppose $[a_n]\to[a]$ and $[b_n]\to[b]$ in the fine topology. Fix $\epsilon>0$ and $t\in\mathbb{N}$. Let N be large enough so that when $n\geq N$ we have

$$\max\left(\sup_k d_H\left(C_{t,k}\left(a_n\right),C_{t,k}(a)\right),\sup_k d_H\left(C_{t,k}\left(b_n\right),C_{t,k}(b)\right)\right)<\frac{\epsilon}{4}. \tag{1}$$

Fix $n\geq N$. Let $k\in\mathbb{N}$ be arbitrary and consider a partition $\mathcal{A}=(A_l)_{l=1}^k$ of X^2 into k pieces. Find partitions $\left(D_i^1\right)_{i=1}^p$ and $\left(D_i^2\right)_{i=1}^q$ of X such that for each $l\leq k$ there are pairwise disjoint sets $I_l\subseteq p\times q$ such that if we write $D_l=\bigcup_{(i,j)\in I_l}D_i^1\times D_j^2$ then

$$\mu^2\left(D_l\triangle A_l\right)<\frac{\epsilon}{4k^2}. \tag{2}$$

Write $(\gamma_s)_{s=1}^t=F$. By (1) we can find a partition $\left(E_i^1\right)_{i=1}^p$ of X such that for all $\gamma\in F$ we have

$$\sum_{i,j=1}^p\left|\mu\left(\gamma^aD_i^1\cap D_j^1\right)-\mu\left(\gamma^{a_n}E_i^1\cap E_j^1\right)\right|<\frac{\epsilon}{4} \tag{3}$$

and a partition $\left(E_i^2\right)_{i=1}^q$ of X such that for all $\gamma\in F$ we have

$$\sum_{i,j=1}^q\left|\mu\left(\gamma^bD_i^2\cap D_j^2\right)-\mu\left(\gamma^{b_n}E_i^2\cap E_j^2\right)\right|<\frac{\epsilon}{4}. \tag{4}$$

Define a partition $\mathcal{B}=(B_l)_{l=1}^k$ of X^2 by setting $B_l=\bigcup_{(i,j)\in I_l}E_i^1\times E_j^2$. For $\gamma\in F$ we now have

$$
\begin{aligned}
&\sum_{l,m=1}^{k} \left|\mu^2(\gamma^{a\times b} D_l \cap D_m) - \mu^2(\gamma^{a_n\times b_n} B_l \cap B_m)\right| \\
&= \sum_{l,m=1}^{k} \left|\mu^2 \left(\gamma^{a\times b}\left(\bigcup_{(i_1,j_1)\in I_l} D^1_{i_1}\times D^2_{j_1}\right)\cap\left(\bigcup_{(i_2,j_2)\in I_m} D^1_{i_2}\times D^2_{j_2}\right)\right)\right. \\
&\qquad\left. - \mu^2\left(\gamma^{a_n\times b_n}\left(\bigcup_{(i_1,j_1)\in I_l} E^1_{i_1}\times E^2_{j_1}\right)\cap\left(\bigcup_{(i_2,j_2)\in I_m} E^1_{i_2}\times E^2_{j_2}\right)\right)\right| \\
&= \sum_{l,m=1}^{k} \left|\mu^2 \left(\left(\bigcup_{(i_1,j_1)\in I_l} \gamma^a D^1_{i_1}\times \gamma^b D^2_{j_1}\right)\cap\left(\bigcup_{(i_2,j_2)\in I_m} D^1_{i_2}\times D^2_{j_2}\right)\right)\right. \\
&\qquad\left. - \mu^2\left(\left(\bigcup_{(i_1,j_1)\in I_l} \gamma^{a_n} E^1_{i_1}\times \gamma^{b_n} E^2_{j_1}\right)\cap\left(\bigcup_{(i_2,j_2)\in I_m} E^1_{i_2}\times E^2_{j_2}\right)\right)\right| \\
&= \sum_{l,m=1}^{k} \left|\mu^2 \left(\bigcup_{\substack{(i_1,j_1,i_2,j_2)\\ \in I_l\times I_m}} \left(\gamma^a D^1_{i_1}\times \gamma^b D^2_{j_1}\right)\cap\left(D^1_{i_2}\times D^2_{j_2}\right)\right)\right. \\
&\qquad\left. - \mu^2\left(\bigcup_{\substack{(i_1,j_1,i_2,j_2)\\ \in I_l\times I_m}} \left(\gamma^{a_n} E^1_{i_1}\times E^2_{j_1}\right)\cap\left(\gamma^{b_n} E^1_{i_2}\times E^2_{j_2}\right)\right)\right| \\
&= \sum_{l,m=1}^{k} \left|\mu^2 \left(\bigcup_{\substack{(i_1,j_1,i_2,j_2)\\ \in I_l\times I_m}} \left(\gamma^a D^1_{i_1}\cap D^1_{i_2}\right)\times\left(\gamma^b D^2_{j_1}\cap D^2_{j_2}\right)\right)\right. \\
&\qquad\left. - \mu^2\left(\bigcup_{\substack{(i_1,j_1,i_2,j_2)\\ \in I_l\times I_m}} \left(\gamma^{a_n} E^1_{i_1}\cap E^1_{i_2}\right)\times\left(\gamma^{b_n} E^2_{j_1}\cap E^2_{j_2}\right)\right)\right| \\
&\le \sum_{l,m=1}^{k} \sum_{\substack{(i_1,j_1,i_2,j_2)\\ \in I_l\times I_m}} \left|\mu\left(\gamma^a D^1_{i_1}\cap D^1_{i_2}\right)\mu\left(\gamma^b D^2_{j_1}\cap D^2_{j_2}\right) - \mu\left(\gamma^{a_n} E^1_{i_1}\cap E^1_{i_2}\right)\mu\left(\gamma^{b_n} E^2_{j_1}\cap E^2_{j_2}\right)\right| \\
&\le \sum_{\substack{(i_1,j_1,i_2,j_2)\\ \in p\times q\times p\times q}} \left|\mu\left(\gamma^a D^1_{i_1}\cap D^1_{i_2}\right)\mu\left(\gamma^b D^2_{j_1}\cap D^2_{j_2}\right) - \mu\left(\gamma^{a_n} E^1_{i_1}\cap E^1_{i_2}\right)\mu\left(\gamma^{b_n} E^2_{j_1}\cap E^2_{j_2}\right)\right| \\
&= \sum_{\substack{(i_1,i_2,j_1,j_2)\\ \in p^2\times q^2}} \left|\mu\left(\gamma^a D^1_{i_1}\cap D^1_{i_2}\right)\mu\left(\gamma^b D^2_{j_1}\cap D^2_{j_2}\right) - \mu\left(\gamma^{a_n} E^1_{i_1}\cap E^1_{i_2}\right)\mu\left(\gamma^{b_n} E^2_{j_1}\cap E^2_{j_2}\right)\right|. \qquad (5)
\end{aligned}
$$

Now (3) and (4) let us apply Lemma 1 with $I = p^2, J = q^2$ and $\delta = \frac{\epsilon}{4}$ to conclude that $(5) \leq \frac{\epsilon}{2}$. Note that for any three subsets S_1, S_2, S_3 of a probability space (Y, ν) we have

$$\begin{aligned}|\nu(S_1 \cap S_3) - \nu(S_2 \cap S_3)| &= |\nu(S_1 \cap S_2 \cap S_3) + \nu((S_1 \setminus S_2) \cap S_3) - \nu(S_1 \cap S_2 \cap S_3) - \nu((S_2 \setminus S_1) \cap S_3)| \\ &\leq \nu(S_1 \triangle S_2),\end{aligned}$$

hence for any $l, m \leq k$ and any action $c \in \mathrm{A}\left(\Gamma, X^2, \mu^2\right)$ we have

$$\begin{aligned}&\left|\mu^2(\gamma^c A_l \cap A_m) - \mu^2(\gamma^c D_l \cap D_m)\right| \\ &\leq \left|\mu^2(\gamma^c A_l \cap A_m) - \mu^2(\gamma^c D_l \cap A_m)\right| + \left|\mu^2\left(\gamma^c D_l \cap A_m\right) - \mu^2\left(\gamma^c D_l \cap D_m\right)\right| \\ &\leq \mu^2\left(\gamma^c A_l \triangle \gamma^c D_l\right) + \mu^2(A_m \triangle D_m) \leq \frac{\epsilon}{2k^2},\end{aligned}$$

where the last inequality follows from (2). Hence for all $\gamma \in F$,

$$\begin{aligned}&\sum_{l,m=1}^{k} \left|\mu^2(\gamma^{a\times b} A_l \cap A_m) - \mu^2(\gamma^{a_n \times b_n} B_l \cap B_m)\right| \\ &\leq \sum_{l,m=1}^{k} \left(\left|\mu^2(\gamma^a A_l \cap A_m) - \mu^2(\gamma^a D_l \cap D_m)\right| + \left|\mu^2(\gamma^{a\times b} D_l \cap D_m) - \mu^2(\gamma^{a_n \times b_n} B_l \cap B_m)\right|\right) \\ &\leq \sum_{l,m=1}^{k} \left(\frac{\epsilon}{2k^2} + \left|\mu^2(\gamma^{a\times b} D_l \cap D_m) - \mu^2(\gamma^{a_n \times b_n} B_l \cap B_m)\right|\right) \\ &\leq \frac{\epsilon}{2} + (5) \leq \epsilon.\end{aligned}$$

Therefore $M^{\mathcal{A}}_{t,k}(a \times b)$ is within ϵ of $M^{\mathcal{B}}_{t,k}(a_n \times b_n)$ and we have shown that for all k, $C_{t,k}(a \times b)$ is contained in the ball of radius ϵ around $C_{t,k}(a_n \times b_n)$. A symmetric argument shows that if $n \geq N$ then for all k, $C_{t,k}(a_n \times b_n)$ is contained in the ball of radius ϵ around $C_{t,k}(a \times b)$ and thus the theorem is proved. □

References

[1] M. Abert and G. Elek. The space of actions, partition metric and combinatorial rigidity. preprint, http://arxiv.org/abs/1108.2147, 2011.

[2] P. Burton. Topology and convexity in the space of actions modulo weak equivalence. preprint, http://arxiv.org/abs/1501.04079.

[3] Alexander S. Kechris. *Global aspects of ergodic group actions*, volume 160 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[4] Robin D. Tucker-Drob. Weak equivalence and non-classifiability of measure preserving actions. *Ergodic Theory Dynam. Systems*, 35(1):293–336, 2015.

Department of Mathematics
The University of Texas at Austin
Austin, TX 78712
pjburton@math.utexas.edu