Productive Lindelöfness and a class of spaces considered by Z. Frolík

November 30, 2011

Peter Burton, Franklin D. Tall[1]

Abstract

We prove that closed subspaces of countable products of σ-compact spaces are productively Lindelöf if and only if there are no Michael spaces. We also prove that, assuming CH, if X is productively Lindelöf and the union of $\aleph_1$ compact sets, then X^ω is Lindelöf.

1 Introduction – Frolík Spaces

On this 50$^{\text{th}}$ anniversary of *TopoSym* it is appropriate to pay tribute to an early participant in these symposia, Zdeněk Frolík, by referring to a paper of his [12] written in the year of the first symposium. We thank Wis Comfort for suggesting we consult [12], which has proved highly relevant to our current research on productively Lindelöf spaces. For convenience, we will assume all spaces in this paper are $T_{3\frac{1}{2}}$.

Definition 1.1. *A space will be called* ***Frolík*** *if it is homeomorphic to a closed subspace of a countable product of σ-compact spaces.*

Rather surprisingly, Frolík [12] proved:

Lemma 1.1. *A space is Frolík if and only if it is $K_{\sigma\delta}$, that is, an intersection of countably many σ-compact subspaces of its Čech-Stone compactification.*

Note that:

Lemma 1.2. *Every Frolík space is powerfully Lindelöf, that is, all of its countable powers are Lindelöf.*

[1] Research supported by NSERC grant A-7354.

(2010) Mathematics Subject Classification. 03E55, 54A25, 54D20, 54D45, 54G20.

Key words and phrases: Alster, Michael space, productively Lindelöf, Rothberger, σ-compact.

This is well-known and follows immediately from the following observation. Consider a Frolík space $F \subseteq \prod_{n<\omega} C_n$, where each C_n is σ-compact. Then F^ω is closed in $\big(\prod_{n<\omega} C_n\big)^\omega$, which is itself a countable product of σ-compact spaces, and hence F^ω is Lindelöf.

Lately we have been investigating powerfully Lindelöf spaces and **productively Lindelöf** spaces, that is, those space X such that $X \times Y$ is Lindelöf for every Lindelöf space Y. Examples of this work include [1], [4], [26], [27] and [25]. An old question of E.A. Michael asks:

Problem 1.1. *Is every productively Lindelöf space powerfully Lindelöf?*

The motivation is:

Lemma 1.3. *Every σ-compact space is powerfully ([12], [16]) and productively Lindelöf.*

Here are some partial results:

Lemma 1.4. *[19] The Continuum Hypothesis (CH) implies that every productively Lindelöf metrizable space is σ-compact.*

Lemma 1.5. *[2] CH implies that every productively Lindelöf space of weight $\leq \aleph_1$ is powerfully Lindelöf.*

Another classic problem of Michael is:

Problem 1.2. *Does there exist a **Michael space**, that is, a Lindelöf space X such that $X \times \mathbb{P}$ is not Lindelöf? Here, $\mathbb{P}$ denotes the space of irrationals. In other words, does $\mathbb{P}$ fail to be productively Lindelöf?*

It is known that (see for example [20]):

Lemma 1.6. $\mathfrak{b} = \aleph_1$ *or* $\mathfrak{d} = Cov(\mathcal{M})$ *implies there is a Michael space.*

Theorem 1.1. *There is no Michael space if and only if every Frolík space is productively Lindelöf.*

Proof. Since the irrationals themselves are a Frolík space, it is clear that there can be no Michael space if Frolík spaces are productively Lindelöf. Thus, let us consider a Lindelöf space L such that $\omega^\omega \times L$ is Lindelöf. It suffices to show that $\prod_{n<\omega} C_n \times L$ is Lindelöf for any sequence $\{C_n\}_{n<\omega}$ of σ-compact spaces. By 3.8.G. in [11], we know that for each n there is a compact space K_n such that C_n can be written as a continuous image of a closed subspace of $\omega \times K_n$. Thus, $\prod_{n<\omega} C_n$ is a continuous image of a closed subspace of $\prod_{n<\omega} \big(\omega \times K_n\big)$, which is homeomorphic to $\omega^\omega \times \prod_{n<\omega} K_n$. But $\prod_{n<\omega} K_n \times L$ is a Lindelöf space, so by assumption $\omega^\omega \times \prod_{n<\omega} K_n \times L$ is a Lindelöf space. So, $\prod_{n<\omega} C_n$ is a continuous image of a closed subspace of a Lindelöf space and we have the result. □

In fact, the above proof gives a slightly sharper statement: if L is a Lindelöf space, and there exists a Frolík space F with $L \times F$ not Lindelöf, then L is a Michael space. The following is another new result.

Theorem 1.2. *Every Frolík space is the union of $\leq \mathfrak{d}$ compact sets, where $\mathfrak{d}$ is the least cardinality of a family of functions cofinal in ω^ω under the $\leq^*$ ordering.*

Proof. Firstly, consider a family $\mathcal{C}$ cofinal in $(\omega^\omega, \leq^*)$. For $f \in \mathcal{C}$ and $n < \omega$, define $f_n : \omega \to \omega$ by $f_n(k) = \max(f(k), n)$. If we take $\mathcal{D} = \{f_n\}_{f\in\mathcal{C}, n<\omega}$, then $\mathcal{D}$ is cofinal in $(\omega^\omega, \leq)$. Moreover, since $\omega < \mathfrak{d} \leq |\mathcal{C}|$, we have $|\mathcal{D}| = |\mathcal{C}| \cdot \omega = |\mathcal{C}|$.

Now, for $n < \omega$, let $C_n = \bigcup_{m<\omega} K_n^m$ be a σ-compact space, where K_n^m is compact. Write $Y = \prod_{n<\omega} C_n$. For $i \in \mathcal{D}$, we define a compact $W_i \subseteq Y$ by $W_i = \prod_{n<\omega} \Big(\bigcup_{k\leq i(n)} K_n^k \Big)$. Claim $Y = \bigcup_{i\in\mathcal{D}} W_i$.

If $y = (y_0, y_1, \ldots) \in Y$, then $y_n \in C_n$ for each $n < \omega$, which implies that for every $n < \omega$ there is a $j(n) < \omega$ such that $y_n \in K_n^{j(n)}$. Then, $y \in \prod_{n<\omega} K_n^{j(n)}$. Choose an $i \in \mathcal{D}$ with $j \leq i$. Then, $\prod_{n<\omega} K_n^{j(n)} \subseteq W_i$, which implies $y \in W_i$ and we have the claim. If $F \subseteq Y$ is closed, then $F \cap W_i$ is compact and $F = \bigcup_{i\in\mathcal{D}} F \cap W_i$. □

Notice that this provides many examples of Lindelöf spaces which are not Frolík.

2 Okunev's Space

There is a Frolík space due to O. Okunev in [3] that has proven to be of considerable interest in our investigations of productive Lindelöfness.

Definition 2.1. *A space X is **Rothberger** if for any sequence $\{\mathcal{U}_n\}_{n<\omega}$ of open covers of X, there are open sets $\{U_n\}_{n<\omega}$ such that $U_n \in \mathcal{U}_n$ and $\bigcup_{n<\omega} U_n = X$. This is the selection principle $\mathsf{S}_1^\omega(\mathcal{O}, \mathcal{O})$. We can also define the corresponding **Rothberger game** $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{O})$ as follows. In the n^{th} round,* ONE *chooses an open cover $\mathcal{U}_n$ and* TWO *chooses a single $U_n \in \mathcal{U}_n$.* TWO *wins if $\{U_n\}_{n<\omega}$ covers X.*

It is a nontrivial result of Pawlikowski [22] that ONE has no winning strategy in the Rothberger game on a space X exactly when X is Rothberger.

Definition 2.2. *A space X is* ***projectively countable*** *if $f(X)$ is countable for every continuous map f from X to a separable metric space.*

Arhangel'skii [3] calls projectively countable spaces ω-**simple**. Note that Lindelöf projectively countable spaces are Rothberger [6].

Example 2.1. *Okunev's space V is formed by taking the Alexandrov duplicate $A(\mathbb{P})$ of the space of irrationals and collapsing the nondiscrete copy of $\mathbb{P}$ to a point. We will let p denote the unique nonisolated point of V, and let q denote the quotient mapping $A(\mathbb{P}) \twoheadrightarrow V$. We will also write $\mathbb{P}_i$ for the copy of $\mathbb{P}$ in $A(\mathbb{P})$ that is homeomorphic to the usual irrationals, and write $\mathbb{P}_d$ for the discrete copy. This construction has the following properties.*

(i) V is $K_{\sigma\delta}$, hence Frolík [3],

(ii) V is not σ-compact [3],

(iii) V is projectively countable [3],

(iv) V is Rothberger,

(v) V does not include a closed copy of $\mathbb{P}$ [27].

Definition 2.3. *[17] A space is* **K**-***analytic*** *if it is the continuous image of a Lindelöf Čech-complete space.*

In [27], we asked whether productively Lindelöf K-analytic spaces must be σ-compact, and in [25] the second author claimed this follows from CH. This is not the case:

(vi) V is K-analytic.

This is immediate from the following.

Theorem 2.1. *[12] If F is a Frolík space, then there is a Čech-complete Frolík space $\tilde{F}$ which maps continuously onto F.*

Another interesting fact about Okunev's space is that since V is $K_{\sigma\delta}$, its growth $V^* = \beta V \setminus V$ is Borel but V^* is not Baire. That is, V^* is an element of the σ-algebra generated by the open sets of βV, but not in the corresponding σ-algebra $\mathcal{Z}$ generated by the zero-sets. To see this, recall that the elements of $\mathcal{Z}$ are Lindelöf (see for example [7]), so supposing $V^* \in \mathcal{Z}$, V would be Lindelöf at infinity. A space is Lindelöf at infinity if and only if every compact set is included in a compact set of countable character [18]. We claim this is a contradiction, since no compact set including the nonisolated point p can be a G_δ. To see this last assertion, suppose $p \in G$, where $G \subseteq V$ is a G_δ. Then $q^{-1}(G) \subseteq A(\mathbb{P})$ is a G_δ, and $\mathbb{P}_i \subseteq G$. Thus, $q^{-1}(G)$ is cocountable, which implies that G is cocountable. But if G were compact, V would be σ-compact.

The second author created unnecessary confusion in [27] by using nonstandard definitions of 'Borel' and 'Baire'. In the same paper we noted that the

Hurewicz Dichotomy does not hold for Okunev's space, since it is not σ-compact nor does it include a closed copy of $\mathbb{P}$. Thus, contrary to [27], the dichotomy does not hold for absolute Borel spaces, but we can ask:

Problem 2.1. *Must every Baire subspace of a compact Hausdorff space either include a closed copy of $\mathbb{P}$ or be σ-compact?*

Using K-analyticity, we can improve Corollary **??**. Note that by Lemma 1.6 as well as the argument given above, $\mathfrak{d} = \aleph_1$ implies there is a Michael space. In [26] it is observed that:

Lemma 2.1. *The existence of a Michael space implies that productively Lindelöf analytic metrizable spaces are σ-compact.*

Definition 2.4. *A space is **projectively σ-compact** if any continuous image in a separable metric space is σ-compact.*

In [23] it is shown that K-analytic metrizable spaces are analytic. Clearly, continuous images of K-analytic spaces are K-analytic, so we can conclude:

Theorem 2.2. *The existence of a Michael space implies that productively Lindelöf Frolík spaces are projectively σ-compact.*

Now, if productively Lindelöf Frolík spaces are projectively σ-compact, then $\mathbb{P}$ is not productively Lindelöf, so there is a Michael space. Rewriting the resulting equivalence, we have:

Corollary 2.1. *There is no Michael space if and only if there is a productively Lindelöf Frolík space which is not projectively σ-compact.*

Corollary 2.2. *There is a productively Lindelöf Frolík space which is not projectively σ-compact if and only if every Frolík space is productively Lindelöf*

Definition 2.5. *A space is **Alster** if whenever each compact set is included in some member of a G_δ cover, then that cover must have a countable subcover.*

(vii) V is Alster and hence ([2]) productively Lindelöf.

Proof. The complement of an open set containing the nonisolated point is Lindelöf and discrete, hence countable. Thus a G_δ containing the nonisolated point is cocountable. It follows that any G_δ cover has a countable subcover. □

Furthermore,

(viii) TWO has a winning strategy for the Rothberger game on V.

Proof. ONE picks the first open cover $\mathcal{U}_0$. Let TWO choose an element $U_0 \in \mathcal{U}_0$ such that $p \in U_0$. Then $\mathbb{P}_i \subseteq q^{-1}(U_0)$. Notice that we can choose $\{x_n\}_{n<\omega}$ such that each $x_n \in \mathbb{P}_i$, and symmetric intervals I_n centered at x_n, so that $\mathbb{P}_i = \bigcup_{n<\omega} I_n$. But since $q^{-1}(U_0)$ is open in $A(\mathbb{P})$, we must then have $(I_n \setminus x_n) \cap \mathbb{P}_d \subseteq$

$q^{-1}(U_0)$. So, no matter what subsequent sequence $\{\mathcal{U}_n\}_{1\leq n<\omega}$ of open covers of V ONE chooses, TWO can pick an element $U_n \in \mathcal{U}_n$ with $q(x_{n-1}) \in U_n$. $\bigcup_{n<\omega} U_n$ is then a cover of V. □

(viii) yields an unusual proof that:

(ix) The nonisolated point p is not a G_δ in V.

This is immediate from the following result of F. Galvin.

Lemma 2.2. *[13]If* TWO *has a winning strategy for the Rothberger game on X and each point of X is a G_δ, then X is countable.*

Lemma 2.2 can also be used to show that TWO winning the Rothberger game is not equivalent to projectively countable.

Theorem 2.3. TWO *having a winning strategy for the Rothberger game implies that a space is projectively countable, but the converse is false.*

Proof. Assume TWO has a winning strategy for the Rothberger game on X, and let $f : X \twoheadrightarrow Y$ map X continuously onto a separable metrizable space Y. Consider the Rothberger game on Y. Any open cover $\mathcal{U}_n$ of Y that ONE chooses gives rise to an open cover $\tilde{\mathcal{U}_n} = \{f^{-1}(U) : U \in \mathcal{U}_n\}$ of X. So, TWO can choose some $f^{-1}(U_n)$ from each $\tilde{\mathcal{U}_n}$ such that $\bigcup_{n<\omega} f^{-1}(U_n) = X$. But then, $\bigcup_{n<\omega} U_n = Y$, so the winning strategy for TWO on X determines a winning strategy on Y. Y is metrizable, thus points of Y are G_δ and Lemma 2.2 implies that Y is countable.

Moore's L-space M in [21] is projectively countable [24]. Hereditarily Lindelöf T_3 spaces have points G_δ, so if TWO had a winning strategy for the Rothberger game on M, Lemma 2.2 would imply that M is countable, which is not the case. □

3 An Application of Elementary Submodels

Another partial result akin to Lemma 1.5 is:

Lemma 3.1. *[27] CH implies every productively Lindelöf space of size $\leq \aleph_1$ is powerfully Lindelöf.*

We can now generalize Lemma 3.1 to obtain:

Theorem 3.1. *CH implies that every productively Lindelöf space which is the union of $\leq \aleph_1$ compact sets is powerfully Lindelöf.*

Proof. This proof is nontrivial, employing a novel application of elementary submodels. Recall that the **Lindelöf number** $L(X)$ of a space X is the least cardinal λ such that every open cover of X has a subcover of size $\leq \lambda$. We straightforwardly generalize the second half of Lemma 1.3 to obtain:

Lemma 3.2. *Suppose X is the union of $\leq \aleph_1$ compact sets. Then $L(X^\omega) \leq \aleph_1$.*

Since X is $T_{3\frac{1}{2}}$, it embeds in a compact space Z. Therefore X^ω embeds in the compact space Z^ω. Write π_n for the projection $Z^\omega \to Z$ and let $X = \bigcup_{\alpha<\omega_1} K_\alpha$, for K_α compact. Then K_α is closed in Z, so $\pi_n^{-1}(K_\alpha)$ is closed in Z^ω. It follows that $\{\pi_n^{-1}(K_\alpha)\}_{n<\omega,\alpha<\omega_1}$ is a family satisfying the hypotheses of the following.

Proposition 3.1. *Let $Y \subseteq Z$ where Z is compact. Suppose there is a family $\{F_\alpha\}_{\alpha<\omega_1}$ of sets closed in Z such that if $x_0 \in Y$ and $x_1 \in Z \setminus Y$, we have $x_0 \in F_{\alpha_0}$ and $x_1 \notin F_{\alpha_0}$ for some $\alpha_0 < \omega_1$. Then $L(Y) \leq \aleph_1$.*

Proof of Proposition 3.1. Let $\mathcal{U} = \{U_\beta\}_{\beta<\kappa}$ be an open cover of Y. Take $V_\beta = Z \setminus \overline{Y \setminus U_\beta}$, where the closure is taken with respect to Z. Since $Y \setminus U_\beta$ is closed in Y, $\overline{Y \setminus U_\beta} \cap Y = \overline{Y \setminus U_\beta}$. This implies $Y \subseteq \bigcup_{\beta<\kappa} V_\alpha = V$. Furthermore, $Y \cap V_\beta \subseteq U_\beta$.

Take $x \in Y$ and note that for each $y \in Z \setminus V$ there is a compact F_{α_y} with $x \in F_{\alpha_y}$ but $y \notin F_{\alpha_y}$. It follows that $\bigcap_{y\in Z\setminus V} F_{\alpha_y} \subseteq V$ and hence $\bigcup_{y\in Z\setminus V} Z \setminus F_{\alpha_y}$ covers $Z \setminus V$, which is compact. Take a finite subcover $\{Z \setminus F_{\alpha_m}\}_{m\leq M}$ of $Z \setminus V$. Then $x \in \bigcap_{m\leq M} F_{\alpha_m} \subseteq V$. This demonstates that, if we let F be the union of all such finite intersections of F_α which meet Y but not $Z \setminus V$, then $Y \subseteq F \subseteq V$.

F is a union of $\aleph_1$ compact sets, so we can take a subcover $\{V_{\beta_\alpha}\}_{\alpha<\omega_1}$. $\{Y \cap V_{\beta_\alpha}\}_{\alpha<\omega_1}$ will then be a refinement of $\mathcal{U}$. □

To prove Theorem 3.1, it then suffices to establish:

Lemma 3.3. *CH implies that if X is productively Lindelöf and $L(X^\omega) \leq \aleph_1$, then X^ω is Lindelöf.*

Proof. In addition to the elementary submodel topology considered in [15], an alternate method of constructing a topology from a space and an elementary submodel containing it is explored in [5], [8] and [10]. Given X and an elementary submodel M with $X \in M$, we define an equivalence relation by letting $x_0 \sim x_1$ for $x_0, x_1 \in X$ if and only if $f(x_0) = f(x_1)$ for every continuous $f : X \to R$ such that $f \in M$. Letting X/M be the resulting quotient and π the projection $X \twoheadrightarrow X/M$, we topologize X/M by taking a base of the form $\pi(U)$, where U is a cozero set in X such that $U \in M$. The basic properties of this construction can be found in any of the papers above, but the most important fact is probably the following.

Lemma 3.4. *[10] For a T_3 space X, X/M is a T_3 space which is a continuous image of X.*

It follows that if X is productively Lindelöf, then so is X/M. Let M be a elementary submodel of size $\leq \aleph_1$ such that $X \in M$. By CH, we can get such an M which is countably closed. Then $w(X/M) \leq \aleph_1$, since X/M has a base of sets which are members of M. By Lemma 1.5, X/M is powerfully Lindelöf.

Now, take an open cover $\mathcal{U}$ of X^ω and assume without loss of generality that $\mathcal{U}$ has size $\aleph_1$. Additionally, assume that every element of $\mathcal{U}$ is basic open of the form $U = \prod_{n<\omega} U_n$, where each U_n is a cozero set in X and cofinitely many $U_n = X$. Assume that $U_n \in M$ for each $n < \omega$ and each $U \in \mathcal{U}$. Consider the map $\Theta : X^\omega \to (X/M)^\omega$ given by reducing each coordinate of a point in X^ω modulo M. More explicitly, if $\pi : X \twoheadrightarrow X/M$ is the quotient map described above, we let $\Theta(x_0, x_1, \ldots) = (\pi(x_0), \pi(x_1), \ldots)$, for $(x_0, x_1, \ldots) \in X^\omega$. I claim that $\Theta^{-1}\Theta(U) = U$ for each $U \in \mathcal{U}$.

Suppose $x = (x_0, x_1, \ldots) \in X^\omega$ and $\Theta(x) = \Theta(y)$, where $y = (y_0, y_1, \ldots) \in U$. If we write $[x_n]$ for the equivalence class $/M$ of a point in X, the statement $\Theta(x) = \Theta(y)$ says $[x_n] = [y_n]$ for every n. By Proposition 2.4.2 in [10], this implies that whenever $H \in M$ is a cozero set, then $x_n \in H$ if and only if $y_n \in H$. But, $y \in U$ implies $y_n \in U_n$ for every n and we assumed $U_n \in M$, hence $x_n \in U_n$ for each M. We have shown $x \in U$, which gives the claim.

So, consider $\{\Theta(U) : U \in \mathcal{U}\}$, which is an open cover of $(X/M)^\omega$. Since X/M is powerfully Lindelöf, there is a countable subcover $\{\Theta(U^k)\}_{k<\omega}$. Pulling this back to $\{\Theta^{-1}\Theta(U^k)\}_{k<\omega} = \{U^k\}_{k<\omega}$ gives a countable subcover of $\mathcal{U}$. This concludes the proof of Lemma 3.3 and hence we have Theorem 3.1. □

This result raises the following question.

Problem 3.1. *If X is productively Lindelöf, is it consistent that $L(X^\omega) \leq 2^{\aleph_0}$?*

This could be combined with CH to solve Problem 1.1. Nothing is known towards an answer except for the following results.

Theorem 3.2. *If X is Lindelöf, $L(X^\omega)$ is less than the first strongly compact cardinal*

Proof. In [9], Drake characterizes the first strongly compact cardinal κ_0 as the least uncountable κ such that if $\mathcal{C}$ is a family of spaces such that every open cover of every $C \in \mathcal{C}$ has a subcover of size $< \kappa$, then every open cover of $\prod \mathcal{C}$ has a subcover of size $< \kappa$. If X is Lindelöf, then clearly $L(X) = \omega < \kappa_0$, so $L(X^\omega) < \kappa_0$. □

This result is notably unsatisfying, since the same argument shows that if X is Lindelöf, then $L(X^\lambda)$ is less than the first strongly compact cardinal for every λ. In terms of possible counterexamples, there is:

Example 3.1. *[14] It is consistent with CH that there is a space X with X^n Lindelöf for every $n < \omega$, but $L(X^\omega) = \aleph_2$.*

The natural attempt to solve Problem 3.1 would be to Lévy-collapse a supercompact to $\aleph_2$ with countable conditions. We do not know what happens in such a model.

References

[1] O. T. Alas, L. F. Aurichi, L. R. Junqueira, and F. D. Tall. Non-productively Lindelöf spaces and small cardinals. *Houston J. Math.*, 37:1373–1381, 2011.

[2] K. Alster. On the class of all spaces of weight not greater than ω_1 whose Cartesian product with every Lindelöf space is Lindelöf. *Fund. Math.*, 129:133–140, 1988.

[3] A. V. Arhangel'skiĭ. Projective σ-compactness, ω_1-caliber, and C_p-spaces. *Topology Appl.*, 104:13–26, 2000.

[4] L. F. Aurichi and F. D. Tall. Lindelöf spaces which are indestructible, productive, or D. *Topology Appl.*, 159(1):331–340, 2011.

[5] I. Bandlow. A construction in set-theoretic topology by means of elementary substructures. *Z. Math. Logik Grundlag. Math.*, 37(5):467–480, 1991.

[6] M. Bonanzinga, F. Cammaroto, and M. Matveev. Projective versions of selection principles. *Topology Appl.*, 157:874–893, 2010.

[7] W.W. Comfort. Remembering Mel Henriksen and (some of) his theorems. *Topology Appl..*, 158(14):1742–1748, 2001.

[8] A. Dow. Set theory in topology. In M. Hušek and J. van Mill, editors, *Recent progress in general topology (Prague, 1991)*, pages 167–197. North-Holland, Amsterdam, 1992.

[9] F.R. Drake. *Set Theory: An Introduction to Large Cardinals.* Studies in Logic and the Foundations of Mathematics (76). North-Holland, Amsterdam, 1984.

[10] T. Eisworth. Elementary submodels and separable monotonically normal compacta. *Top. Proc.*, 30(2):431–443, 2006.

[11] R. Engelking. *General Topology.* Heldermann Verlag, Berlin, 1989.

[12] Z. Frolík. On the descriptive theory of sets. *Czechoslovak Math. J.*, 20:335–359, 1963.

[13] F. Galvin. Indeterminacy of point-open games. *Bull. Acad. Polon. Sci. Sr. Sci. Math. Astronom. Phys.*, 26(5):445–449, 1978.

[14] I. Gorelic. On powers of Lindelöf spaces. *Comment. Math. Univ. Carolin.*, 35(2):383–401, 1994.

[15] R. Grunberg, L.R. Junqueira, and F.D. Tall. Forcing and normality. *Topology Appl.*, 84:145–174, 1998.

[16] A.W. Hager. Approximations of continuous real-valued functions on Lindelöf spaces. *Proc. Amer. Math. Soc.*, 22:156–163, 1969.

[17] R.W. Hansell. Descriptive topology. In M. Hušek and J. van Mill, editors, *Recent progress in general topology (Prague, 1991)*, pages 275–315. North-Holland, Amsterdam, 1992.

[18] M. Henriksen and J.R. Isbell. Some properties of compactifications. *Duke Math. J.*, 25:83–105, 1957.

[19] E. A. Michael. Paracompactness and the Lindelöf property in finite and countable Cartesian products. *Compositio Math.*, 23:199–214, 1971.

[20] J. T. Moore. Some of the combinatorics related to Michael's problem. *Proc. Amer. Math. Soc.*, 127:2459–2467, 1999.

[21] J.T. Moore. A solution to the L-space problem. *J. Amer. Math. Soc.*, 19:717–736, 2006.

[22] J. Pawlikowski. Undetermined sets of point-open games. *Fund. Math.*, 144:279–285, 1994.

[23] C.A. Rogers and J.E. Jayne et al. *Analytic Sets.* Academic Press, London, 1980.

[24] M. Scheepers and F. D. Tall. Lindelöf indestructibility, topological games and selection principles. *Fund. Math.*, 210:1–46, 2010.

[25] F. D. Tall. Productively Lindelöf spaces may all be D. Canad. Math. Bulletin, in press, 2010.

[26] F. D. Tall. Lindelöf spaces which are Menger, Hurewicz, Alster, productive, or D. *Topology Appl.*, 158(18):2556–2563, 2011.

[27] F. D. Tall and B. Tsaban. On productively Lindelöf spaces. *Topology Appl.*, 158(11):1239–1248, 2011.

peter.burton@utoronto.ca
Department of Mathematics, University of Toronto
40 St. George Street, Toronto, Ontario, Canada M5S 2E4

f.tall@utoronto.ca
Department of Mathematics, University of Toronto
40 St. George Street, Toronto, Ontario, Canada M5S 2E4