Flexible stability and nonsoficity

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June 7, 2019

Abstract

A sofic group G is said to be flexibly stable if every sofic approximation to G can converted to a sequence of disjoint unions of Schreier graphs by modifying an asymptotically vanishing proportion of edges. We establish that if $\mathrm{PSL}_d(\mathbb{Z})$ is flexibly stable for some $d \geq 5$ then there exists a group which is not sofic.

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1 Introduction

1.1 Sofic groups

Soficity is a finite approximation property for countable discrete groups which has received considerable attention in recent years. A group is called sofic if it admits a sofic approximation, which is a sequence of partial actions on finite sets that asymptotically approximates the action of the group on itself by left-translations. The precise definition appears below. Soficity can be thought of as a common generalization of amenability and residual finiteness. We refer the reader to [6, 13] for surveys.

It is a famous open problem to determine whether every countable discrete group is sofic. It is also widely open to classify sofic approximations to well-known groups, for example by showing that every sofic approximation is asymptotically equivalent to an approximation by actions on finite sets (as opposed to partial actions). If a group has this latter property, it is called flexibly stable. The main result of this paper is that if $\mathrm{PSL}_d(\mathbb{Z})$ is flexibly stable for some $d \geq 5$ then there is a nonsofic group. The proof gives an explicit group G, constructed as a quotient of an HNN-extension of $\mathrm{PSL}_d(\mathbb{Z})$, that is not sofic if $\mathrm{PSL}_d(\mathbb{Z})$ is flexibly stable.

We now formulate precise definitions to state the result.

Definition 1.1. *Let G be a countable discrete group. A **sofic approximation** to G consists of a sequence $(V_n)_{n=1}^{\infty}$ of finite sets and a sequence $(\sigma_n)_{n=1}^{\infty}$ of functions $\sigma_n : G \to \mathrm{Sym}(V_n)$ such that the following conditions hold, where we write σ_n^g instead of $\sigma_n(g)$.*

- *Asymptotic homomorphisms: For every fixed pair $g, h \in G$ we have*

$$\lim_{n\to\infty} \frac{1}{|V_n|} |\{v \in V_n : \sigma_n^g(\sigma_n^h(v)) = \sigma_n^{gh}(v)\}| = 1.$$

- *Asymptotic freeness: For every fixed nontrivial element $g \in G$ we have*

$$\lim_{n\to\infty} \frac{1}{|V_n|} |\{v \in V_n : \sigma_n^g(v) = v\}| = 0.$$

*We say that G is **sofic** if there exists a sofic approximation to G.*

1.2 Flexible stability

Definition 1.2. *A sofic approximation* $(\sigma_n : G \to \mathrm{Sym}(V_n))_{n=1}^{\infty}$ *is* ***perfect*** *if each* σ_n *is a genuine group homomorphism.*

If S is a finite generating set for G we can endow V_n with the structure of a S-labelled directed graph by putting an s-labelled edge from v to $\sigma^s(v)$ for each $s \in S$ and $v \in V_n$. Accordingly, we refer to the V_n as the vertex sets of the sofic approximation. With this structure, each connected component of a perfect sofic approximation to G is a Schreier graph on the cosets of a finite-index subgroup of G.

Definition 1.3. *Let* $\Sigma = (\sigma_n : G \to \mathrm{Sym}(V_n))_{n=1}^{\infty}$ *and* $\Xi = (\xi_n : G \to \mathrm{Sym}(V_n))_{n=1}^{\infty}$ *be two sofic approximations to* G *with the same vertex sets. We say that* Σ *and* Ξ *are at* ***edit-distance zero*** *if for each fixed* $g \in G$ *we have*

$$\lim_{n\to\infty} \frac{1}{|V_n|} |\{v \in V_n : \sigma^g(v) = \xi^g(v)\}| = 1.$$

Now suppose the vertex sets of $\Xi = (\xi_n : G \to \mathrm{Sym}(W_n))_{n=1}^{\infty}$ *are not necessarily the same as the vertex sets of* Σ*. We say that* Σ *and* Ξ *are* ***conjugate*** *if there exist finite sets* U_n *and injections* $\pi_n : V_n \to U_n$*,* $\rho_n : W_n \to U_n$ *such that*

$$1 = \lim_{n\to\infty} \frac{|V_n|}{|U_n|} = \lim_{n\to\infty} \frac{|W_n|}{|U_n|}$$

and such that the sofic approximations $(\pi_{n*}\sigma_n)_{n=1}^{\infty}$ *and* $(\rho_{n*}\xi_n)_{n=1}^{\infty}$ *are at edit-distance zero. Here* $\pi_{n*}\sigma_n : G \to \mathrm{Sym}(U_n)$ *is the map defined by:*

$$(\pi_{n*}\sigma_n)^g(\pi_n(v)) = \pi_n(\sigma_n^g(v))$$

for $v \in V_n$ *and*

$$(\pi_{n*}\sigma_n)^g(u) = u$$

if $u \in U_n \setminus \pi_n(V_n)$*. The map* $\rho_{n*}\xi_n : G \to \mathrm{Sym}(U_n)$ *is defined similarly.*

Definition 1.4. *We say that a sofic group* G *is* ***flexibly stable*** *if every sofic approximation to* G *is conjugate to a perfect sofic approximation to* G*.*

It is clear that a flexibly stable group is residually finite. It is also clear that free groups are flexibly stable. In [10] it is shown that surface groups are flexibly stable. A group G is said to be **strictly**

stable if every sofic approximation is conjugate to a perfect sofic approximation where the conjugacies π and ρ as in Definition 1.3 are bijections. In [2] it is shown that finitely generated abelian groups are strictly stable. In [4] it is shown that polycyclic groups are strictly stable. In [3] it is shown that no infinite property (T) group is strictly stable. The most elementary example for which flexible stability is unknown seems to be the direct product of the rank two free group with $\mathbb{Z}$.

The main result of this paper is the following.

Theorem 1.1. *Suppose that* $\mathrm{PSL}_d(\mathbb{Z})$ *is flexibly stable for some* $d \geq 5$*. Then there exists a group which is not sofic.*

The nonsofic group of the theorem has the following form. Let H be a countable discrete group with subgroups A and B and suppose there is an isomorphism $\phi : A \to B$. The **HNN extension** $H*_\phi$ is defined to be $(H * \langle t \rangle)/N$ where $H * \langle t \rangle$ is the free product of H with a copy of $\mathbb{Z}$ and N is the smallest normal subgroup of $H * \langle t \rangle$ containing all elements of the form $tat^{-1}\phi(a)^{-1}$ for $a \in A$. We will need a mod 2 version of the construction above. So let N_2 be the smallest normal subgroup of $H * \langle t \rangle$ containing all elements of the form $tat^{-1}\phi(a)^{-1}$ for $a \in A$ along with t^2. Let $H *_\phi /2 = (H * \langle t \rangle)/N_2$.

In Section 2, we show that if H is flexibly stable and if H, A, B and ϕ satisfy certain technical conditions then the group $H *_\phi /2$ cannot be sofic. This part of the argument is completely general in that it does not use anything specific to $\mathrm{PSL}_d(\mathbb{Z})$. The rest of the paper involves constructing two subgroups A and B of $\mathrm{PSL}_d(\mathbb{Z})$ and showing that they possess the required properties. This part uses a ping-pong type argument that originates in the reference [1]. Other precursors to this idea can be found in work on strong approximation in [11], on maximal subgroups of $\mathrm{PSL}_d(\mathbb{Z})$ in [9] and on the congruence subgroup property in [12]. We need that $d \geq 5$ only because this condition guarantees that all $\mathrm{PSL}_2(\mathbb{Z})$ orbits in $\mathrm{PSL}_d(\mathbb{Z}/p\mathbb{Z})$ have density bounded by a constant which is strictly less than 1. We do not know whether the result can be improved to $d \in \{3, 4\}$.

Because Theorem 1.1 uses such heavy machinery, it is natural to wonder whether results of its type can be found among other groups. For example, if H is a direct product of two free groups then do there exist subgroups A and B satisfying the criteria of Theorem 2.1? What if H is a lattice in the isometry group of quaternionic hyperbolic space? Another interesting case would be to establish Theorem 2.1 for a 2-Kazhdan group such as a higher-rank p-adic lattice. The relevance of this last case is that in [8] it is shown that 2-Kazhdan groups satisfy the analog of flexible stability for homomorphisms into finite-dimensional unitary groups with the unnormalized Frobenius metric. It is unknown whether $\mathrm{PSL}_d(\mathbb{Z})$ is 2-Kazhdan.

1.3 Acknowledgments

The first author would like to thank Tsachik Gelander for conversations on this problem many years ago, including a proof that certain groups constructed in a manner similar to the group appearing in the main theorem are not residually finite and therefore are good candidates for being nonsofic. We thank Emmanuel Breuillard for helpful discussions related to the proof of Lemma 3.4. We also thank Yves Stalder for catching some errors in a previous version.

2 General results

Theorem 2.1. *Suppose H is a flexibly stable countable discrete group with subgroups A and B satisfying the following conditions.*

(1) If $K \leq H$ has finite index, then every B-orbit in H/K is contained in an A-orbit. Explicitly, this means for every $h \in H$ we have $BhK \subseteq AhK$.

(2) If C is the subgroup generated by A and B then there is an automorphism $\omega \in \mathrm{Aut}(C)$ such that $\omega(A) = B$ and ω^2 is the identity.

(3) There is a constant $\lambda > 1$ such that if K is a proper finite index subgroup of H then for every $g, h \in H$ we have

$$|AgK| \geq \lambda |BhK| \tag{2.1}$$

where the cardinality $|\cdot|$ is taken in H/K.

(4) A has property (τ) with respect to the family of finite index subgroups

$$\{K \cap A : \; K \leq H, [H : K] < \infty\}$$

Then the group

$$G = \langle H, t | t^2 = 1, tat^{-1} = \omega(a) \; \forall a \in A \rangle$$

is not sofic.

The proof of Theorem 2.1 is in Subsection 2.3 below after some preliminaries.

2.1 Property (τ)

This section reviews Property (τ).

Definition 2.1. *Let $\Gamma = (V, E)$ be a finite graph. If $W \subseteq V$ the* ***edge boundary*** *in Γ of W will be denoted $\partial_\Gamma W$ and consists of all edges $(v, w) \in E$ where $v \in W$ and $w \notin W$. If W is nonempty the* ***edge isoperimetric ratio*** *of W will be denoted $\iota_\Gamma(W)$ and is defined to be $|\partial_\Gamma W| \, |W|^{-1}$. The* ***edge expansion constant*** *of Γ will be denoted $e(\Gamma)$ and is defined to be the minimum value of $\iota_\Gamma(E)$ over all nonempty subsets $W \subseteq V$ satisfying $|W| \leq \frac{1}{2}|V|$.*

Definition 2.2. *Let $(\Gamma_n)_{n=1}^\infty$ be a sequence of finite connected graphs and let $c > 0$. We say that $(\Gamma_n)_{n=1}^\infty$ forms a family of c-**expanders** if $\inf_{n \in \mathbb{N}} e(\Gamma_n) \geq c$. We say that $(\Gamma_n)_{n=1}^\infty$ forms a family of* ***expanders*** *if it forms a family of c-expanders for some $c > 0$.*

Definition 2.3. *Let G be a group, $H \leq G$ and $S \subset G$. The* **Schreier coset graph** Schreier$(G/H, S)$ *is the multi-graph with vertex set G/H and edges $\{gH, sgH\}$ for all $gH \in G/H$ and $s \in S$. Multiple edges and self-loops are allowed.*

Definition 2.4. *A group G has* **Property (τ) with respect to a family** *$\mathcal{F}$ of finite index subgroups of G if there is a finite generating set $S \subset G$ and a constant $c > 0$ such that for every $H \in \mathcal{F}$ we have that* Schreier$(G/H, S)$ *is a c-expander.*

It is easy to see that Property (τ) for a family $\mathcal{F}$ is does not depend on the choice of S.

2.2 Modular HNN extensions

Let H be a countable discrete group with subgroups $A, B \leq H$ and suppose there is an isomorphism $\phi : A \to B$. The **HNN extension** $H*_\phi$ is defined to be $(H * \langle t \rangle)/N$ where $H * \langle t \rangle$ is the free product of H with a copy of $\mathbb{Z}$ and N is the smallest normal subgroup of $H * \langle t \rangle$ containing all elements of the form $tat^{-1}\phi(a)^{-1}$ for $a \in A$. We will need a mod 2 version of the construction above. So let N_2 be the smallest normal subgroup of $H * \langle t \rangle$ containing all elements of the form $tat^{-1}\phi(a)^{-1}$ for $a \in A$ along with t^2. Let $H *_\phi /2 = (H * \langle t \rangle)/N_2$.

Lemma 2.1. *Let C be the subgroup of H generated by A and B. Assume there exists an automorphism ω of C such that ω^2 is the identity and such that $\omega(a) = \phi(a)$ for all $a \in A$ and $\omega(b) = \phi^{-1}(b)$ for all $b \in B$. Then the canonical homomorphism from H to $H *_\phi /2$ is injective.*

Proof of Lemma 2.1. Let D be the semidirect product $C \rtimes \mathbb{Z}/2\mathbb{Z}$ where $\mathbb{Z}/2\mathbb{Z}$ acts on C via the automorphism τ. We claim that $H *_\phi /2$ can be constructed as the free product of H with D amalgamated over the common subgroup C. Indeed, $H *_C D$ is naturally generated by H and the additional generator $t = t^{-1}$ of $\mathbb{Z}/2\mathbb{Z}$. If $a \in A$ then tat is equal to $\omega(a) = \phi(a)$ and similarly if $b \in B$ then tbt is equal to $\omega(b) = \phi^{-1}(b)$. Therefore $tat\phi(a)^{-1}$ and $tbt\phi^{-1}(b)^{-1}$ are trivial in $H *_C D$ for all $a \in A$ and all $b \in B$. By the universal property of free products with amalgamation we see that these relations suffice to describe $H *_C D$ and so we have established the claim. Since the factor groups always inject into an amalgamated free product this completes the proof of Lemma 2.1. □

2.3 Proof of Theorem 2.1

We now prove Theorem 2.1. By Lemma 2.1, the canonical homomorphism from H into G is injective. Thus we identify H as a subgroup of G from now on. Assume toward a contradiction that there exists a sofic approximation $\Sigma = (\sigma_n : G \to \mathrm{Sym}(V_n))_{n=1}^\infty$ to G. Since H is flexibly stable, we may assume without loss of generality that the restriction of Σ to H is perfect.

Since A has property (τ) with respect to the family

$$\{K \cap A : \; K \leq H, [H : K] < \infty\}$$

there exists a finite generating set $S \subset A$ and a constant $c > 0$ such that for every finite index subgroup K of H all connected components of the Schreier coset graph Schreier$(H/K, S)$ are c-expanders. Let Γ_n be the graph on V_n corresponding to $\{\sigma_n^s : s \in S\}$. Explicitly, this means that the edges of Γ_n are the pairs $\{v, \sigma_n^s(v)\}$ for $v \in V_n$ and $s \in S$. By the above remarks, every connected component of Γ_n is a c-expander.

Let Λ_n be the graph on V_n corresponding to $\{\sigma_n^{\omega(s)} : s \in S\}$. The hypothesis that every B-orbit is contained in an A-orbit implies that every Λ_n-connected component is contained in a Γ_n-connected component.

For the remainder of Subsection 2.3 we fix $n \in \mathbb{N}$ such that σ_n is a sufficiently good sofic approximation for certain conditions stated later to hold. We suppress the subscript n in notations.

Let $\Omega_1, \ldots, \Omega_m$ be an enumeration of the connected components of Γ such that $|\Omega_j| \geq |\Omega_{j+1}|$ for all $j \in \{1, \ldots, m-1\}$. Let D be the set of all $w \in V$ such that $w \in \Omega_j$ and $\sigma^t(w) \in \Omega_k$ where $j \leq k$. If σ is a sufficiently good sofic approximation then for at least $\frac{9}{10}|V|$ vertices $w \in V$ we must have that $(\sigma^t)^2(w) = w$. If the last condition is satisfied then at least one of w and $\sigma^t(w)$ is an element of D. Therefore $|D| \geq \frac{9}{20}|V|$.

Let $\mathcal{I} \subseteq \{1, \ldots, m\}$ be the set of all indexes j such that $|D \cap \Omega_j| \geq \frac{1}{10}|\Omega_j|$. We must have

$$\sum_{j \in \mathcal{I}} |\Omega_j| \geq \frac{|V|}{10}. \tag{2.2}$$

Fix $j \in \mathcal{I}$ and consider the set $D \cap \Omega_j$. Let $\Theta_1, \ldots, \Theta_r$ be the partition of $D \cap \Omega_j$ into the intersections of $D \cap \Omega_j$ with σ^t-pre-images of connected components of Λ. Let $q \in \{1, \ldots, r\}$ and suppose $\sigma^t(\Theta_q) \subseteq \Omega_k$. Since $\sigma^t(\Theta_q)$ is contained in a single connected component of Λ we see from (2.1) that $\lambda|\sigma^t(\Theta_q)| \leq |\Omega_k|$. Since $\Theta_q \subseteq D$ we have $j \leq k$ so that $|\Omega_j| \geq |\Omega_k|$ and therefore $\lambda|\Theta_q| \leq |\Omega_j|$ which implies $(\lambda - 1)|\Theta_q| \leq |\Omega_j \setminus \Theta_q|$.

Since $\Theta_q \subset \Omega_j$ we have $\partial_\Gamma \Theta_q = \partial_\Gamma(\Omega_j \setminus \Theta_q)$. Since every connected component of Γ is a c-expander,

$$|\partial_\Gamma \Theta_q| \geq c \min\left(|\Theta_q|, |\Omega_j \setminus \Theta_q|\right) \geq c \min(1, \lambda - 1)|\Theta_q|.$$

Let $c' = c \min(1, \lambda - 1)$. So

$$|\partial_\Gamma \Theta_1 \cup \cdots \cup \partial_\Gamma \Theta_r| \geq \frac{c'}{2}(|\Theta_1| + \cdots + |\Theta_r|) = \frac{c'}{2}|D \cap \Omega_j| \geq \frac{c'}{20}|\Omega_j| \tag{2.3}$$

Here the first inequality holds because the pairwise disjointness of the Θ_q guarantees that for any edge e there are at most two indices q such that $e \in \partial_\Gamma \Theta_q$.

Let $q \in \{1, \ldots, r\}$, let $v \in \Theta_q$ and suppose (v, w) is an edge in $\partial \Theta_q$. If $w \notin D$ then $\sigma^t(v)$ and $\sigma^t(w)$ are in different connected components of Γ, and so in particular they are in different connected components of Λ. On the other hand, if $w \in D$ then by hypothesis $\sigma^t(v)$ and $\sigma^t(w)$ are in different connected components of Λ. Hence in either case $(\sigma^t(v), \sigma^t(w))$ is not an edge in Λ.

From (2.3) we see that for at least $\frac{c'}{20}|\Omega_j|$ edges (v, w) in $\Gamma \upharpoonright \Omega_j$ the image $(\sigma^t(v), \sigma^t(w))$ is not an edge in Λ. Summing (2.3) over all $j \in \mathcal{I}$ we see from (2.2) that there is a set K of edges in Γ with $|K| \geq \frac{c'}{200}|V|$ such that for each $(v, w) \in K$ the image $(\sigma^t(v), \sigma^t(w))$ is not an edge in Λ. However, if σ is a sufficiently good sofic approximation then the number of such edges should be an arbitrarily small fraction of $|V|$. Thus we have obtained a contradiction and the proof of Theorem 2.1 is complete.

3 Subgroups of special linear groups

In Section 3 we will prove that $\mathrm{PSL}_d(\mathbb{Z})$ satisfies the conditions of Theorem 2.1 for $d \geq 5$, thereby completing the proof of Theorem 1.1

3.1 Ping-pong arguments

The next Lemma constructs the subgroups A and B that will be used in our application of Theorem 2.1.

Lemma 3.1. *Let $d \geq 3$. Identify $\mathrm{PSL}_2(\mathbb{Z})$ as a subgroup of $\mathrm{PSL}_d(\mathbb{Z})$ by the homomorphism*

$$M \in \mathrm{PSL}_2(\mathbb{Z}) \mapsto \left(\begin{array}{c|c} M & 0 \\ \hline 0 & I_{d-2} \end{array}\right) \in \mathrm{PSL}_d(\mathbb{Z}).$$

Then there exist subgroups A and B of $\mathrm{PSL}_d(\mathbb{Z})$ such that the following hold.

(1) A and B are free groups of rank 4,

(2) A is profinitely dense in $\mathrm{PSL}_d(\mathbb{Z})$,

(3) B is contained in $\mathrm{PSL}_2(\mathbb{Z})$ and

(4) the subgroup $\langle A, B\rangle$ is free of rank 8.

Proof of Lemma 3.1. By the main theorem of [1] there exists a profinitely dense free subgroup A of $\mathrm{PSL}_d(\mathbb{Z})$ with rank 4. The construction of this subgroup gives additional information about A that we will use. To describe this, we recall the following notions from [1].

An element $g \in \mathrm{PSL}_d(\mathbb{Z})$ is **hyperbolic** if it is semisimple, admits a unique (counting multiplicities) eigenvalue of maximal absolute value and a unique eigenvalue of minimum absolute value. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of generalized eigenvectors such that v_1 corresponds to the unique maximal eigenvalue of g and v_n corresponds to the unique minimal eigenvalue. Let $\alpha(g) = [v_1] \in \mathbb{RP}^{d-1}$ and $\rho(g) = [\mathrm{span}(v_2, \ldots, v_n)] \subset \mathbb{RP}^{d-1}$. These are the attracting fixed point and repelling hyperplane of g. Note that $\alpha(g^{-1}) = [v_n]$ and $\rho(g^{-1}) = [\mathrm{span}(v_1, \ldots, v_{n-1})]$. Although g need not be diagonalizable, $\rho(g)$ does not depend on the choice of basis $\{v_1, \ldots, v_n\}$.

Definition 3.1. *Let $g_0, g_1, \ldots, g_s \in \mathrm{PSL}_d(\mathbb{Z})$ be hyperbolic elements. Then $\{g_1, \ldots, g_s\}$ is a g_0-***rooted free system** *if there exist open sets $O_i \subset \mathbb{RP}^{d-1}$ for $i \in \{0, 1, \ldots, s\}$ such that the following hold.*

(1) The sets $\{O_i\}_{i=0}^s$ are pairwise disjoint,

(2) for all $j \in \{0, \ldots, s\}$ we have

$$\alpha(g_j) \cup \alpha(g_j^{-1}) \subseteq O_i \subseteq \overline{O_i} \subseteq \mathbb{RP}^{d-1} \setminus (\rho(g_0) \cup \rho(g_0^{-1}))$$

(3) $\alpha(g_0) \cup \alpha(g_0^{-1}) \subset O_0 \subseteq \overline{O_0} \subset \mathbb{RP}^{d-1} \setminus \left(\bigcup_{j=1}^{s} \rho(g_i) \cup \rho(g_j^{-1}) \right)$,

(4) *and* $\bigcup_{n \in \mathbb{Z} \setminus \{0\}} g_j^n(\overline{O_j}) \subset O_k$ *for all distinct pairs* $j, k \in \{0, \ldots, s\}$.

The standard ping-pong lemma from [14] shows that if $\{g_1, \ldots, g_s\}$ form a g_0-rooted free system then $\{g_0, \ldots, g_s\}$ freely generate a free group of rank $s+1$. The construction in [1] shows that there exist hyperbolic elements $g_0, g_1, g_2, g_3, g_4 \in \mathrm{PSL}_d(\mathbb{Z})$ such that $\{g_1, g_2, g_3, g_4\}$ is a g_0-rooted free system and the subgroup $\langle g_1, g_2, g_3, g_4 \rangle$ is profinitely dense. (The definition of g_0-rooted free system that we use differs slightly from the one used in [1] . However, it is easy to verify that their proof gives a g_0-rooted free system in our sense.) We make the following claim.

Claim 3.1. *After conjugating the elements above if necessary, we may assume that* $\rho(g_j) \cup \rho(g_j^{-1})$ *does not contain* $[span(e_1, e_2)]$ *and* $\alpha(g_j) \cup \alpha(g_j^{-1})$ *is not contained in* $[span(e_3, e_4, e_5)]$ *for any* $j \in \{0, \ldots, 4\}$.

Proof of Claim 3.1. Let V_j be the set of all $h \in \mathrm{PSL}_d(\mathbb{R})$ such that $h(\rho(g_j) \cup \rho(g_j^{-1}))$ does not contain $[\mathrm{span}(e_1, e_2)]$. Let W_j be the set of all $h \in \mathrm{PSL}_d(\mathbb{R})$ such that $\alpha(g_j) \cup \alpha(g_j^{-1})$ is not contained in $[\mathrm{span}(e_3, e_4, e_5)]$. Then both V_j and W_j are Zariski-open and nonempty. Since $\mathrm{PSL}_d(\mathbb{R})$ is Zariski-connected and $\mathrm{PSL}_d(\mathbb{Z})$ is Zariski-dense, the set

$$\mathrm{PSL}_d(\mathbb{Z}) \cap \left(\bigcap_{j=0}^{4} (V_j \cap W_j) \right) \tag{3.1}$$

is non-empty. Let h be an element of the set in (3.1). Replacing each g_j with hg_jh^{-1} proves Claim 3.1. $\square$

It is well-known that given any finite subset F of $\mathbb{RP}^1$, there exists a hyperbolic element $f \in \mathrm{PSL}_2(\mathbb{Z})$ which has no fixed point in F. Using Claim 1, this implies the existence of hyperbolic elements $h_1, h_2, h_3, h_4 \in \mathrm{PSL}_2(\mathbb{Z})$ such that the each of following sets is empty.

$$\left(\bigcup_{j=1}^{4} \{\alpha(h_j), \alpha(h_j^{-1})\} \right) \cap \left(\bigcup_{k=0}^{4} (\rho(g_k) \cup \rho(g_k^{-1})) \right)$$
$$\left(\bigcup_{j=1}^{4} \{\rho(h_j), \rho(h_j^{-1})\} \right) \cap \left(\bigcup_{k=0}^{4} (\alpha(g_k) \cup \rho(a_k^{-1})) \right)$$

$$\{\alpha(h_j), \alpha(h_j^{-1})\} \cap \{\alpha(h_k), \alpha(h_k^{-1})\} \tag{3.2}$$

Here, the set in (3.2) should be empty for all distinct pairs $j, k \in \{1, 2, 3, 4\}$.

Now, let $\{O_i\}_{i=0}^4$ be open sets witnessing the statement that $\{g_1, \ldots, g_4\}$ is a g_0-rooted free system. After replacing g_0 with g_0^n for some $n \in \mathbb{N}$ if necessary, we may replace O_0 with a smaller open neighborhood and thereby obtain

$$O_0 \cap \left(\bigcup_{j=1}^{4} (\rho(h_j) \cup \rho(h_j^{-1})) \right) = \emptyset. \tag{3.3}$$

Let $N_0 \in \mathbb{N}$ be large enough so that

$$\bigcup_{|n| \geq N_0} g_0^n \left(\bigcup_{j=1}^{4} (\alpha(h_j) \cup \alpha(h_j^{-1})) \right) \subset O_0.$$

Choose open sets U_j for $j \in \{1, \ldots, 4\}$ such that the following hold.

- We have
$$\{\alpha(h_j), \alpha(h_j^{-1})\} \subseteq U_j \subseteq \overline{U_i} \subseteq \mathbb{RP}^{d-1} \setminus (\rho(g_0) \cup \rho(g_0^{-1})$$
- we have $O_0 \cap U_j = U_j \cap U_k = \emptyset$ for all distinct pairs $j, k \in \{1, \ldots, 4\}$
- and we have
$$\bigcup_{|n| \geq N_0} \bigcup_{j=1}^{4} g_0^n U_j \subseteq O_0$$

Let $n_0 \geq N_0$ and let $U_0 \subset O_0$ be an open neighborhood of $\{\alpha(g_0), \alpha(g_0^{-1})\}$ such that we have

$$\emptyset = U_0 \cap \left(\bigcup_{j=1}^{4} g_0^{-n_0} \left(O_j \cup \rho(g_j) \cup \rho(g_j^{-1}) \right) \right) \tag{3.4}$$

and $g_0^{n_0} \overline{U_0} \subseteq O_0$. Choose $N_1 \in \mathbb{N}$ large enough so that

$$\bigcup_{|n|\geq N_1}\bigcup_{j=1}^{4} g_0^{-n}(\overline{O_j}\cup\overline{U_j})\subseteq U_0. \tag{3.5}$$

Let $n_1 \geq n_0 + N_1$. Finally, choose $N_2 \in \mathbb{N}$ large enough so that

$$\bigcup_{|m|\geq N_2}\bigcup_{\ell\neq j}\bigcup_{k=1}^{4} h_j^m(g_0^{-n_0}\overline{O_k}\cup\overline{U_0}\cup\overline{U_\ell})\subseteq U_j \tag{3.6}$$

for all $j \in \{1,\ldots,4\}$. After replacing h_j with $h_j^{N_2}$ for all $j \in \{1,\ldots,4\}$, we may assume $N_2 = 1$. Set $h_0 = g_0^{n_1}$, $h_{4+j} = g_0^{-n_0}g_jg_0^{n_0}$ and $U_{4+j} = g_0^{-n_0}O_j$ for $j \in \{1,\ldots,4\}$. We claim that the sets $\{U_j\}_{j=0}^8$ witness the fact that $\{h_j\}_{i=1}^8$ is an h_0-rooted free system. Proposition 3.1 will follow by setting $A = \langle h_5, h_6, h_7, h_8\rangle$ and $B = \langle h_1, h_2, h_3, h_4\rangle$.

To verify Condition (1) in Definition 3.1, let $0 \leq j < k \leq 8$. We must show $U_j \cap U_k = \emptyset$. We consider five cases.

- Suppose $j = 0$ and $k \leq 4$. Since $U_0 \subset O_0$ and $O_0 \cap U_k = \emptyset$, $U_0 \cap U_k = \emptyset$.
- Suppose $j = 0$ and $k > 4$. Then $U_0 \cap U_k = U_0 \cap g_0^{-n_0}O_{k-4} = \emptyset$ by (3.4).
- Suppose $1 \leq j < k \leq 4$. Then U_j and U_k are disjoint by our choice of U_j and U_k.
- Suppose $1 \leq j \leq 4 < k \leq 8$. Then we have
$$U_j \cap U_k = g_0^{-n_0}\left(g_0^{n_0}U_j \cap O_{k-4}\right) \subset g_0^{-n_0}(O_0 \cap O_{k-4}) = \emptyset$$
- Finally, suppose $4 < i < j \leq 8$. We have $U_j = g_0^{-n_0}O_{j-4}$ and $U_k = g_0^{-n_0}O_{k-4}$ and $O_{j-4}\cap O_{k-4} = \emptyset$ by assumption.

Thus we have verified Condition (1) in Definition 3.1

Now, let $j \in \{0,\ldots,8\}$. To verify Condition (2) in Definition 3.1, we must show that

$$\alpha(h_j)\cup\alpha(h_j^{-1})\subseteq U_j\subseteq\overline{U_j}\subseteq\mathbb{RP}^{d-1}\setminus(\rho(g_0)\cup\rho(g_0^{-1})) \tag{3.7}$$

If $j = 0$ this follows from $U_0 \subset O_0$ and the corresponding fact about O_0. For $j \in \{1, \dots, 4\}$, this follows from the choice of U_j. For $j \in \{5, \dots, 8\}$ we have

$$\alpha(h_j) \cup \alpha(h_j^{-1}) = g_0^{-n_0}[\alpha(g_{j-4}) \cup \alpha(g_{j-4}^{-1})]$$

Since $U_j = g_0^{-n_0} O_{j-4}$, this implies the first two inclusions in (3.7). The last inclusion in (3.7) is equivalent to

$$\overline{g_0^{-n_0} O_{j-4}} \subseteq \mathbb{RP}^{d-1} \setminus (\rho(g_0) \cup \rho(g_0^{-1}))$$

which holds since $O_{j-4} \subseteq \mathbb{RP}^{d-1} \setminus (\rho(g_0) \cup \rho(g_0^{-1}))$. Thus we have verified Condition (2) in Definition 3.1.

Condition (3) in Definition 3.1 follows immediately from the following observations.

- We have $U_0 \subset O_0$.
- The set O_0 is disjoint from $\bigcup_{j=1}^4 \rho(h_j) \cup \rho(h_j^{-1})$ by (3.3).
- The set U_0 is disjoint from $\bigcup_{j=1}^4 g_0^{-n_0}[O_j \cup \rho(g_j) \cup \rho(g_j^{-1})]$ by (3.4).

To verify Condition (4) in Definition 3.1 let $j, k \in \{0, \dots, 8\}$ be distinct. We must check that

$$\bigcup_{m \in \mathbb{Z} \setminus \{0\}} h_i^m(\overline{U_k}) \subset U_j$$

If $j = 0$ then this follows from

$$\begin{aligned} \bigcup_{k=1}^{8} \bigcup_{m \in \mathbb{Z} \setminus \{0\}} h_0^m(\overline{U_k}) &= \bigcup_{k=1}^{8} \bigcup_{m \in \mathbb{Z} \setminus \{0\}} g_0^{n_1 m}(\overline{U_k}) \\ &= \bigcup_{k=1}^{4} \bigcup_{m \in \mathbb{Z} \setminus \{0\}} g_0^{n_1 m}(\overline{U_k} \cup g_0^{-n_0}\overline{O_k}) \end{aligned}$$

and (3.5). If $j \in \{1, \dots, 4\}$, then Condition (4) follows from (3.6) since $N_2 = 1$. Finally, let $j \in \{5, \dots, 8\}$. If also $k \in \{5, \dots, 8\}$, then since the O_j's witness that $\{g_1, \dots, g_4\}$ is a g_0-rooted free system we have

$$\bigcup_{m \in \mathbb{Z} \setminus \{0\}} h_j^m(\overline{U_k}) = \bigcup_{m \in \mathbb{Z} \setminus \{0\}} g_0^{-n_0} g_j^m g_0^{n_0} g_0^{-n_0}(\overline{O_k})$$

$$
\begin{aligned}
&= \bigcup_{m\in\mathbb{Z}\setminus\{0\}} g_0^{-n_0} g_j^m(\overline{O_k}) \\
&\subseteq g_0^{-n_0} O_j = U_j
\end{aligned}
$$

If $k \in \{0, \ldots, 4\}$ then by our choice of n_0 and U_0 we have

$$
\begin{aligned}
\bigcup_{m\in\mathbb{Z}\setminus\{0\}} h_j^m(\overline{U_k}) &= \bigcup_{m\in\mathbb{Z}\setminus\{0\}} g_0^{-n_0} g_j^m g_0^{n_0}(\overline{U_k}) \\
&\subseteq \bigcup_{m\in\mathbb{Z}\setminus\{0\}} g_0^{-n_0} g_j^m O_0 \\
&\subseteq g_0^{-n_0} O_j = U_j
\end{aligned}
$$

This completes the verification of Condition (4) and thereby completes the proof of Lemma $\square$

3.2 Expansion in quotients of $\mathrm{PSL}_d(\mathbb{Z})$

Lemma 3.2. *Let $d \geq 3$. Let A be a profinitely dense subgroup of $\mathrm{PSL}_d(\mathbb{Z})$. Then A has property (τ) with respect to the family*

$$\{K \cap A : \ K \leq \mathrm{PSL}_d(\mathbb{Z}), [\mathrm{PSL}_d(\mathbb{Z}) : K] < \infty\}$$

Proof. Because A is profinitely dense, it is Zariski dense. Let $S \subset A$ be a finite generating set. Theorem 1 in [5] asserts that the Cayley graphs of $\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z})$ with respect to S form a family of c-expanders for some $c > 0$. Let $K \leq \mathrm{PSL}_d(\mathbb{Z})$ have finite index. By the congruence subgroup property as established in [12], there exists a $q \in \mathbb{N}$ such that K contains the kernel Γ_q of the natural surjection

$$\mathrm{PSL}_d(\mathbb{Z}) \twoheadrightarrow \mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z}).$$

It follows that the quotient map $\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z}) \twoheadrightarrow \mathrm{PSL}_d(\mathbb{Z})/K$ induces a covering space

$$\mathrm{Schreier}(\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z}), S) \twoheadrightarrow \mathrm{Schreier}(\mathrm{PSL}_d(\mathbb{Z})/K, S)$$

Therefore the preimage of a subset D of $\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z})/K$ has the same edge isoperimetric ratio as D. Since $\mathrm{Schreier}(\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z}), S)$ is a c-expander, so is $\mathrm{Schreier}(\mathrm{PSL}_d(\mathbb{Z})/K, S)$. $\square$

3.3 Bounds on the density of $\mathrm{PSL}_2(\mathbb{Z})$-orbits in finite quotients of $\mathrm{PSL}_d(\mathbb{Z})$

The main result of this section is Lemma 3.4, which provides an upper bound on densities of $\mathrm{PSL}_2(\mathbb{Z})$-orbits in finite quotients of $\mathrm{PSL}_d(\mathbb{Z})$. First we prove a lemma that allows us to reduce the general case to the $\mathrm{PSL}_d(\mathbb{Z}/p\mathbb{Z})$ case.

Lemma 3.3. *Let $q \in \mathbb{N}$ and let $K \leq \mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z})$ be a proper subgroup. Then there is a prime factor p of q such that the image of K under reduction* mod p *is a proper subgroup of* $\mathrm{PSL}_d(\mathbb{Z}/p\mathbb{Z})$.

Proof. It suffices to consider the special case in which K is a maximal proper subgroup. Suppose toward a contradiction that the proposition fails for K. We may assume without loss of generality that q has the minimal number of distinct prime factors among all $r \in \mathbb{N}$ such that the proposition fails for some subgroup of $\mathrm{PSL}_d(\mathbb{Z}/r\mathbb{Z})$.

Recall that if G is a finite group then the Frattini subgroup $\Phi(G)$ is the intersection of all maximal proper subgroups of G. If G and H are finite groups we have $\Phi(G \times H) = \Phi(G) \times \Phi(H)$. Let $q = p_1^{n_1} \cdots p_k^{n_k}$ be the prime factorization of q. By the Chinese remainder theorem, we have that $\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z})$ is isomorphic to $\mathrm{PSL}_d(\mathbb{Z}/p_1^{n_1}\mathbb{Z}) \times \cdots \times \mathrm{PSL}_d(\mathbb{Z}/p_k^{n_k}\mathbb{Z})$. Since the Frattini subgroup is normal and $\mathrm{PSL}_d(\mathbb{Z}/p\mathbb{Z})$ is simple, we have that $\mathrm{PSL}_d(\mathbb{Z}/p_j^{n_j}\mathbb{Z})/\Phi(\mathrm{PSL}_d(\mathbb{Z}/p_j^{n_j}\mathbb{Z}))$ is isomorphic to $\mathrm{PSL}_d(\mathbb{Z}/p_j\mathbb{Z})$. Therefore $\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z})/\Phi(\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z}))$ is isomorphic to

$$\mathrm{PSL}_d(\mathbb{Z}/p_1\mathbb{Z}) \times \cdots \times \mathrm{PSL}_d(\mathbb{Z}/p_k\mathbb{Z}). \tag{3.8}$$

Since $\Phi(\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z})) \leq K$ we may assume without loss of generality that $n_j = 1$ for all $j \in \{1, \ldots, k\}$.

Let G_j be the product in (3.8) where the j^{th} factor is replaced by the trivial group. Write π_j for the projection from $\mathrm{PSL}(\mathbb{Z}/q\mathbb{Z})$ onto G_j. If $\pi_j(K)$ is a proper subgroup of G_j then by the minimality assumption on the number of prime factors of q we see that the projection of $\pi_j(K)$ to some factor $\mathrm{PSL}_d(\mathbb{Z}/p_m\mathbb{Z})$ for $m \in \{1, \ldots, k\} \setminus \{j\}$ is not surjective. Thus we may assume that $\pi_j(K) = G_j$ for all $j \in \{1, \ldots, k\}$.

For each $j \in \{1, \ldots, k\}$ the intersection $K \cap G_j$ is a normal subgroup of K. Since π_j is surjective from K to G_j we see that $\pi_j(K \cap G_j)$ is a normal subgroup of G_j. Since $\pi_j(K \cap G_j) = K \cap G_j$ we obtain that $K \cap G_j$ is a normal subgroup of G_j. Since each group $\mathrm{PSL}_d(\mathbb{Z}/p_j\mathbb{Z})$ is simple we obtain that $K \cap G_j$ is equal to a product

$$\prod_{m \in S_j} \mathrm{PSL}(\mathbb{Z}/p_m\mathbb{Z})$$

for a set $S_j \subseteq \{1, \ldots, k\} \setminus \{j\}$. (We regard this product as a subset of $\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z})$ by replacing the missing factors with the trivial group.) If $S_j = \{1, \ldots, k\} \setminus \{j\}$ then since K is a proper subgroup of $\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z})$ we must have that the projection of K onto $\mathrm{PSL}_d(\mathbb{Z}/p_j\mathbb{Z})$ is not surjective and thus we are done in this case. Therefore we may assume that S_j is a proper subset of $\{1, \ldots, k\} \setminus \{j\}$.

Let $S = \bigcup_{j=1}^{k} S_j$. Then K contains the product

$$\prod_{m \in S} \mathrm{PSL}_d(\mathbb{Z}/p_m\mathbb{Z}) \tag{3.9}$$

and $K \cap \mathrm{PSL}_d(\mathbb{Z}/p_m\mathbb{Z})$ is trivial when $m \notin S$. After passing to the quotient of $\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z})$ by the subgroup in (3.9) we reduce to the case when $K \cap G_j$ is trivial for all $j \in \{1, \ldots, k\}$. However, $K \cap G_j$ is the kernel of the projection from $\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z})$ onto $\mathrm{PSL}_d(\mathbb{Z}/p_j\mathbb{Z})$. Thus we obtain that K is isomorphic to $\mathrm{PSL}_d(\mathbb{Z}/p_j\mathbb{Z})$ for all j. This contradiction completes the proof of Lemma 3.3. $\square$

Lemma 3.4. *Let $d \geq 5$ and let $\Sigma = (\sigma_n : \mathrm{PSL}_d(\mathbb{Z}) \to \mathrm{Sym}(W_n))_{n=1}^{\infty}$ be a perfect sofic approximation to $\mathrm{PSL}_d(\mathbb{Z})$. Let B be a subgroup of the copy of $\mathrm{PSL}_2(\mathbb{Z})$ in the upper left corner of $\mathrm{PSL}_d(\mathbb{Z})$. Then for all sufficiently large n the maximal size of a $\sigma_n(B)$-orbit in W_n is at most $\frac{1}{16}$ the size of the $\sigma_n(\mathrm{PSL}_d(\mathbb{Z}))$-orbit which contains it.*

Proof. It suffices to consider the case when the action of $\mathrm{PSL}_d(\mathbb{Z})$ on W_n is transitive. Thus we may assume that $W_n = \mathrm{PSL}_d(\mathbb{Z})/H_n$ for finite-index subgroups $(H_n)_{n=1}^{\infty}$ of $\mathrm{PSL}_d(\mathbb{Z})$ and σ_n is the left-translation action. Using the congruence subgroup property we see that it suffices to show that if $q \geq 2$ then for any proper subgroup K of $\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z})$ the maximal size of a $\mathrm{PSL}_2(\mathbb{Z}/q\mathbb{Z})$-orbit in $\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z})/K$ is at most $\frac{1}{16}|\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z})/K|$.

Using Lemma 3.3 we see that there exists a prime factor p of q such that if we write π for the projection of $\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z})$ onto $\mathrm{PSL}_d(\mathbb{Z}/p\mathbb{Z})$ then $\pi(K)$ is a proper subgroup of $\mathrm{PSL}_d(\mathbb{Z}/p\mathbb{Z})$. The map π sends $\mathrm{PSL}_2(\mathbb{Z})$-orbits in $\mathrm{PSL}_d(\mathbb{Z}/q\mathbb{Z})$ to $\mathrm{PSL}_2(\mathbb{Z})$-orbits in $\mathrm{PSL}_d(\mathbb{Z}/p\mathbb{Z})$. Moreover is m-to-1 for some fixed m. Therefore it suffices to show that if L is a proper subgroup of $\mathrm{PSL}_d(\mathbb{Z}/p\mathbb{Z})$ for some prime p then the maximal size of a $\mathrm{PSL}_2(\mathbb{Z})$-orbit in $\mathrm{PSL}_d(\mathbb{Z}/p\mathbb{Z})/L$ is at most $\frac{1}{16}|\mathrm{PSL}_d(\mathbb{Z}/p\mathbb{Z})|$.

The $\mathrm{PSL}_2(\mathbb{Z})$-orbits in $\mathrm{PSL}_d(\mathbb{Z}/p\mathbb{Z})/L$ are the double cosets $\mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})xL$ for $x \in \mathrm{PSL}_d(\mathbb{Z}/p\mathbb{Z})$. In [7] it is shown that the maximal size of a proper subgroup of $\mathrm{PSL}_d(\mathbb{Z}/p\mathbb{Z})$ for a prime p is $(p^d-1)(p-1)^{-1}$. For any $d \in \mathbb{N}$ we have

$$|\mathrm{PSL}_d(\mathbb{Z}/p\mathbb{Z})| = \frac{1}{\gcd(d, p-1)(p-1)} \prod_{j=0}^{d-1}(p^d - p^j)$$

so that in particular

$$|\mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})| = \frac{(p^2-p)(p^2-1)}{\gcd(d, p-1)(p-1)}.$$

Therefore if $d \geq 5$ we have

$$\begin{aligned}
\frac{|\mathrm{PSL}_d(\mathbb{Z}/p\mathbb{Z})|}{|\mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})|\,|L|} &= \frac{1}{|L|}\frac{(p^d-p)(p^d-1)}{(p^2-p)(p^2-1)}\prod_{j=2}^{d-1}(p^d-p^j) && (3.10)\\
&\geq \frac{1}{|L|}\prod_{j=2}^{d-1}(p^d-p^j) && (3.11)\\
&\geq \frac{p-1}{p^d-1}\prod_{j=2}^{d-1}(p^d-p^j) \\
&= \frac{(p^d-p^2)(p^d-p^3)}{p^d-1}(p-1)\prod_{j=4}^{d-1}(p^d-p^j) && (3.12)\\
&\geq (p-1)\prod_{j=4}^{d-1}(p^d-p^j) && (3.13)\\
&\geq 16
\end{aligned}$$

Here, (3.11) follows from (3.10) and (3.13) follows from (3.12) since in each case the factor dropped is at least one. It follows that any double coset $\mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})xL$ has size at most $\frac{1}{16}|\mathrm{PSL}_d(\mathbb{Z}/p\mathbb{Z})|$ and so the proof of Lemma 3.4 is complete. $\square$

Theorem 1.1 is obtained by applying Theorem 2.1 to the subgroups A and B constructed in Proposition 3.1. Because A is profinitely dense, it surjects onto every finite quotient. In particular, every B-orbit in a finite quotient of $\mathrm{PSL}_d(\mathbb{Z})$ is contained in an A-orbit. To define the automorphism $\omega : C \to C$, let A be freely generated by $\{a_1, a_2, a_3, a_4\}$ and B be freely generated by $\{b_1, b_2, b_3, b_4\}$. Then C is freely generated by $\{a_j, b_j\}_{j=1}^4$. So there is a unique order 2 automorphism defined by $\omega(a_j) = b_j$ and $\omega(b_j) = a_j$ for $j \in \{1, \ldots, 4\}$. By Lemmas 3.2 and 3.4 the subgroups A and B satisfy the other conditions of Theorem 2.1.

References

[1] Menny Aka, Tsachik Gelander, and Gregory A. Soĭfer. Homogeneous number of free generators. *J. Group Theory*, 17(4):525–539, 2014.

[2] Goulnara Arzhantseva and Liviu Păunescu. Almost commuting permutations are near commuting permutations. *J. Funct. Anal.*, 269(3):745–757, 2015.

[3] Oren Becker and Alexander Lubotzky. Group stability and Property (T). *arXiv e-prints*, page arXiv:1809.00632, Sep 2018.

[4] Oren Becker, Alexander Lubotzky, and Andreas Thom. Stability and Invariant Random Subgroups. *arXiv e-prints*, page arXiv:1801.08381, Jan 2018.

[5] Jean Bourgain and Péter P. Varjú. Expansion in $SL_d(\mathbf{Z}/q\mathbf{Z})$, q arbitrary. *Invent. Math.*, 188(1):151–173, 2012.

[6] Valerio Capraro and Martino Lupini. *Introduction to Sofic and hyperlinear groups and Connes' embedding conjecture*, volume 2136 of *Lecture Notes in Mathematics*. Springer, Cham, 2015. With an appendix by Vladimir Pestov.

[7] Bruce N. Cooperstein. Minimal degree for a permutation representation of a classical group. *Israel J. Math.*, 30(3):213–235, 1978.

[8] Marcus De Chiffre, Lev Glebsky, Alex Lubotzky, and Andreas Thom. Stability, cohomology vanishing, and non-approximable groups. *arXiv e-prints*, page arXiv:1711.10238, Nov 2017.

[9] T. Gelander and C. Meiri. Maximal subgroups of $\mathrm{SL}(n, \mathbb{Z})$. *Transform. Groups*, 21(4):1063–1078, 2016.

[10] Nir Lazarovich, Arie Levit, and Yair Minsky. Surface groups are flexibly stable. *arXiv e-prints*, page arXiv:1901.07182, Jan 2019.

[11] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991.

[12] Jens L. Mennicke. Finite factor groups of the unimodular group. *Ann. of Math. (2)*, 81:31–37, 1965.

[13] Vladimir G. Pestov. Hyperlinear and sofic groups: a brief guide. *Bull. Symbolic Logic*, 14(4):449–480, 2008.

[14] J. Tits. Free subgroups in linear groups. *J. Algebra*, 20:250–270, 1972.

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