

# Invariant random subgroups and action versus representation maximality

Peter J. Burton and Alexander S. Kechris

## 1 Introduction

Let  $G$  be a countably infinite group and  $(X, \mu)$  a standard non-atomic probability space. We denote by  $A(G, X, \mu)$  the space of measure preserving actions of  $G$  on  $(X, \mu)$  with the weak topology. If  $\mathbf{a}, \mathbf{b} \in A(G, X, \mu)$ , we say that  $\mathbf{a}$  is **weakly contained** in  $\mathbf{b}$ , in symbols  $\mathbf{a} \preceq \mathbf{b}$ , if  $\mathbf{a}$  is in the closure of the set of isomorphic copies of  $\mathbf{b}$  (i.e., it is in the closure of the orbit of  $\mathbf{b}$  under the action of the automorphism group of  $(X, \mu)$  on  $A(G, X, \mu)$ ; see [K]). We say that  $\mathbf{a} \in A(G, X, \mu)$  is **action-maximal** if for all  $\mathbf{b} \in A(G, X, \mu)$  we have  $\mathbf{b} \preceq \mathbf{a}$ . Such  $\mathbf{a}$  exist by a result of Glasner-Thouvenot-Weiss, Hjorth, see [K, Theorem 10.7]).

Now let  $H$  be a separable, infinite-dimensional Hilbert space and denote by  $\text{Rep}(G, H)$  the space of unitary representations of  $G$  on  $H$  with the weak topology (see [K, Appendix H]). For  $\pi, \rho \in \text{Rep}(G, H)$  we denote by  $\pi \preceq \rho$  the usual relation of **weak containment** of representations (see [BHV], [K, Appendix H]). We say that  $\pi \in \text{Rep}(G, H)$  is **representation-maximal** if for all  $\rho \in \text{Rep}(G, H)$  we have  $\rho \preceq \pi$ . It is easy to check that such  $\pi$  exist.

For any action  $\mathbf{a} \in A(G, X, \mu)$ , let  $\kappa^{\mathbf{a}}$  be the associated representation on  $L^2(X, \mu)$ , called the **Koopman representation**, and by  $\kappa_0^{\mathbf{a}}$  its restriction to the orthogonal of the constant functions (see [K, page 66]). Then we have

$$\mathbf{a} \preceq \mathbf{b} \implies \kappa_0^{\mathbf{a}} \preceq \kappa_0^{\mathbf{b}}$$

but the converse fails, see [K, pages 66 and 68] and also [CK, page 155] for examples. However in all these examples the actions  $\mathbf{a}, \mathbf{b}$  were not both ergodic and this led to the following question.

---

Research partially supported by NSF Grant DMS-1464475

**Problem 1.1.** *If  $\mathbf{a}, \mathbf{b} \in A(G, X, \mu)$  are free, ergodic, does  $\kappa_0^{\mathbf{a}} \preceq \kappa_0^{\mathbf{b}}$  imply  $\mathbf{a} \preceq \mathbf{b}$ ?*

We provide a negative answer below. The proof is based on a result about invariant random subgroups of  $G = \mathbf{F}_\infty$ , the free group on a countably infinite set of generators, which might be of independent interest.

If  $I$  is a countable set and  $\alpha$  is an action of a countable group  $G$  on  $I$ , we will write  $\mathbf{s}_\alpha$  for the corresponding **generalized shift action** on  $2^I$  with the usual product measure, given by  $(\mathbf{s}_\alpha(g) \cdot f)(i) = f(\alpha(g)^{-1} \cdot i)$ . If  $I = G/H$ , for some  $H \leq G$ , we will write  $\tau_{G/H}$  for the left-translation action of  $G$  on  $G/H$  and  $\mathbf{s}_{G/H}$  instead of  $\mathbf{s}_{\tau_{G/H}}$ . If  $H$  is trivial, we write  $\mathbf{s}_G$  instead of  $\mathbf{s}_{G/H}$ .

We also let  $\lambda_\alpha$  be the representation on  $\ell^2(I)$  given by  $(\lambda_\alpha(g) \cdot f)(i) = f(\alpha(g)^{-1} \cdot i)$ . Note that  $\lambda_{\tau_{G/H}}$  is the usual **quasi-regular representation** of  $G$  on  $\ell^2(G/H)$ , which we will denote by  $\lambda_{G/H}$ .

We call a subgroup  $H \leq G$  with  $[G : H] = \infty$  **action-maximal** if  $\mathbf{s}_{G/H}$  is action-maximal and **representation-maximal** if  $\lambda_{G/H}$  is representation-maximal. It was shown in [K1] that there are  $H$  which are action-maximal and also  $H$  which are representation-maximal, for any non-abelian free group  $G$ .

An **invariant random subgroup (IRS)** of  $G$  is a probability Borel measure on  $\text{Sub}(G)$ , the compact space of subgroups of  $G$ , which is invariant under the (continuous) action of  $G$  on  $\text{Sub}(G)$  by conjugation. Denote by  $\mathcal{M}_G \subseteq \text{Sub}(G)$  the set of all  $H \leq G$  that are both action-maximal and representation-maximal. We show the following:

**Theorem 1.1.** *Let  $G = \mathbf{F}_\infty$ . Then there exists an IRS of  $G$  which is supported by  $\mathcal{M}_G$ .*

Using this and the result of Dudko-Grigorchuk [DG, Proposition 8], we then prove the following:

**Theorem 1.2.** *Let  $G = \mathbf{F}_\infty$ . Then there exists a free, ergodic  $\mathbf{a} \in A(G, X, \mu)$  such that  $\mathbf{a}$  is not action-maximal but  $\kappa_0^{\mathbf{a}}$  is representation-maximal.*

Let  $\mathbf{a}$  be as in Theorem 1.2. Since  $G = \mathbf{F}_\infty$  does not have property (T), the free, ergodic actions  $\mathbf{b} \in A(G, X, \mu)$  are dense in  $A(G, X, \mu)$  (see [K, Theorems 12.2 and 10.8]), so there is a free, ergodic  $\mathbf{b} \in A(G, X, \mu)$  such that  $\mathbf{b} \not\preceq \mathbf{a}$ . On the other hand  $\kappa_0^{\mathbf{b}} \preceq \kappa_0^{\mathbf{a}}$ , thus we have a negative answer to Problem 1.1.

We employ below the following notation:

If  $\alpha$  is an action of  $G$  on  $I$  and  $S \subseteq G$ , we write  $\alpha(S) = \{\alpha(g) : g \in S\} \subseteq \text{Sym}(I)$ . For  $G = \mathbf{F}_\infty$ , we let  $g_0, g_1, \dots$  be free generators of  $G$  and let  $G_n = \langle g_0, g_1, \dots, g_n \rangle \leq G$ .

If  $x$  is a real number, we write  $\lfloor x \rfloor$  for the largest integer less than or equal to  $x$ . If  $x, y$  are real numbers and  $\epsilon > 0$ , we write  $x \approx_\epsilon y$  to mean  $|x - y| < \epsilon$ . Finally,  $\mathbb{N} = \{0, 1, 2, \dots\}$  and  $\mathbb{N}^+ = \{1, 2, 3, \dots\}$ .

**For the rest of the paper,  $G = \mathbf{F}_\infty$ .**

## 2 Proof of Theorem 1.1

The structure of the proof is as follows. In Subsection 2.1 we state three lemmas. Temporarily assuming these lemmas, in Subsection 2.2 we give the main argument establishing Theorem 1.1. Then in Subsection 2.3 we prove the lemmas from Subsection 2.1.

Recall that for  $\mathbf{a} \in A(G, X, \mu)$ , we have  $\mathbf{a} \preceq \mathbf{b}$  if and only if  $\mathbf{a}$  lies in the closure of the isomorphic copies of  $\mathbf{b}$ . In particular,  $\mathbf{b}$  is action-maximal if and only if the isomorphic copies of  $\mathbf{b}$  are dense in  $A(G, X, \mu)$ . We will use these equivalences without comment several times in the sequel.

### 2.1 Statements of lemmas

The first lemma provides a general method for constructing invariant random subgroups.

**Lemma 2.1.** *Let  $\alpha$  be an action of  $G$  on a countably infinite set  $I$ . Suppose there is an increasing sequence of non-empty finite subsets  $(F_n)_{n=0}^\infty$  of  $I$  such that  $\bigcup_{n=0}^\infty F_n = I$  and  $F_n$  is  $\alpha(G_n)$ -invariant. Let  $\theta_n$  be the probability measure on  $\text{Sub}(G)$  given by the pushforward of the uniform measure on  $F_n$  under the map  $v \mapsto \text{stab}_\alpha(v)$  (where  $\text{stab}_\alpha(v)$  is the stabilizer of  $v$  in  $\alpha$ ). Let  $\theta$  be any weak-star limit point of the  $\theta_n$ . Then  $\theta$  is an invariant random subgroup of  $G$ .*

In order to state the second lemma, we need the following definition.

**Definition 2.1.** *Let  $\alpha$  be an action of  $G$  on a finite set  $V$  and let  $n$  be such that all  $\alpha(g_k), k > n$ , act trivially. Let  $\beta$  be an action of  $G$  on a countably infinite set  $I$ . Let  $Q \subseteq I$  be a finite set. We will say that  $\alpha$  (relative to  $n$ ) **appears in  $\beta$  within  $Q$**  if there is a  $\beta(G_n)$ -invariant set  $W \subseteq Q$  and a bijection  $\phi : V \rightarrow W$  such that  $\phi(\alpha(g) \cdot v) = \beta(g) \cdot \phi(v)$  for all  $v \in V$  and*

$g \in G_n$ . We will say that  $\alpha$  **appears in**  $\beta$  if it appears within some finite subset of  $I$ .

Note that if  $\alpha$  appears in  $\beta$  as above, then  $\mathbf{s}_{\alpha|G_n}$  is a factor of  $\mathbf{s}_{\beta|G_n}$ .

**Lemma 2.2.** *There exists a sequence of finite sets  $(V_n)_{n=1}^{\infty}$ , with  $|V_n| \rightarrow \infty$ , and actions  $(\alpha_n)_{n=1}^{\infty}$  of  $G$ , where  $\alpha_n$  acts transitively on  $V_n$  so that all  $g_k, k > n$ , act trivially in  $\alpha_n$ , such that if  $\beta$  is a transitive action of  $G$  on a countably infinite set and  $\alpha_n$  (relative to  $n$ ) appears in  $\beta$  for each  $n$ , then  $\mathbf{s}_{\beta}$  is action-maximal and  $\lambda_{\beta}$  is representation-maximal.*

Fix a sequence of finite sets  $V_n$  and actions  $\alpha_n$  of  $G$  on  $V_n$ ,  $n \geq 1$ , as in Lemma 2.2. Given  $f : \mathbb{N} \rightarrow \mathbb{N}^+$ ,  $m > 0$ , write  $C_m(f) = \sum_{n=0}^{m-1} (|V_{f(n)}| + 1)$ . We will need a function  $f$  with the following properties.

**Lemma 2.3.** *There exists a function  $f : \mathbb{N} \rightarrow \mathbb{N}^+$  such that:*

- (i) *for every  $n \geq 1$  there exists positive integer  $K = K_n$  such that for all  $j$  there is  $l$  with  $\lfloor \frac{j}{K} \rfloor = \lfloor \frac{l}{K} \rfloor$  and  $f(l) = n$ ,*
- (ii) *for every  $\epsilon > 0$ , there exists  $t > 0$ , such that for all  $m > 0$  we have*

$$\frac{1}{C_m(f)} \sum_{n=1}^t (|V_n| + 1) \cdot |\{j \in \{0, \dots, m-1\} : f(j) = n\}| > 1 - \epsilon.$$

## 2.2 Main argument

Let, for  $n \geq 1$ ,  $\alpha_n$  and  $V_n$  be as in Lemma 2.2 and let  $f$  be as in Lemma 2.3. Choose a pairwise disjoint sequence of finite sets  $W_n$ ,  $n \geq 0$ , such that  $|W_n| = |V_{f(n)}|$ . Define an action of  $\alpha$  of  $G$  on  $\bigcup_{n=0}^{\infty} W_n$  by identifying  $W_n$  with  $V_{f(n)}$  and letting  $G$  act on  $W_n$  according to  $\alpha_{f(n)}$ . Let  $\{u_n\}_{n=0}^{\infty}$  be an enumeration of a countably infinite set disjoint from the  $W_n$ . We now modify  $\alpha$  to obtain a new action  $\beta$  of  $G$  on  $I = \left(\bigcup_{n=0}^{\infty} W_n\right) \cup \{u_n\}_{n=0}^{\infty}$ . We will have that  $\beta(g_k)$  agrees with  $\alpha(g_k)$  on  $W_n$  when  $k \in \{0, \dots, f(n)\}$ .

For each  $n$ , choose a point  $w_n \in W_n$  and let  $\beta(g_{f(n)+1})$  transpose  $w_n$  with  $u_n$ . Let  $(l_n)_{n=0}^{\infty}$  be a strictly increasing sequence of indices such that  $\max(n, f(0), \dots, f(n+1)) + 1 < l_n$ . Let  $\beta(g_{l_n})$  transpose  $w_n$  and  $w_{n+1}$ .

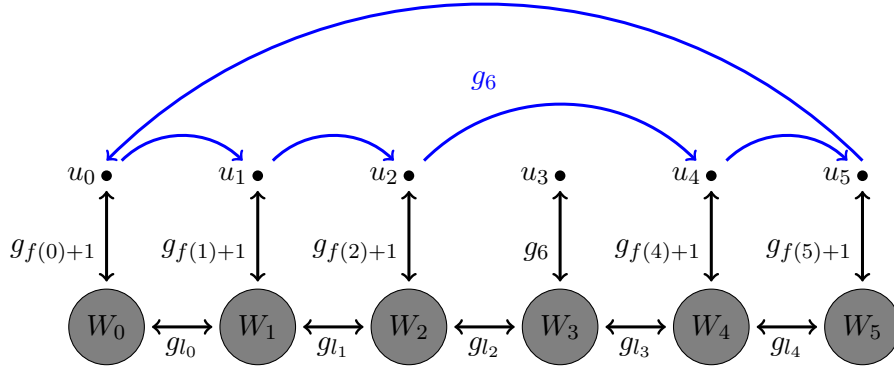
Fix  $n \geq 1$ . We now define how  $\beta(g_n)$  acts on  $\{u_j\}_{j=0}^\infty$ . For  $k \in \mathbb{N}$ , consider the discrete interval

$$D_{n,k} = \{k \cdot n, \dots, (k+1) \cdot n - 1\}.$$

We would like to have  $\beta(g_n)$  make a cycle out  $\{u_j, j \in D_{n,k}\}$  for each  $k$ . Unfortunately, we cannot achieve that exactly since there may be some  $j \in D_{n,k}$  for which  $f(j) + 1 = n$ , and in this case we will have already used  $g_n$  to link  $W_j$  with  $u_j$ . Thus for each  $k$ , we will let  $\beta(g_n)$  make a cycle out of the set

$$\{u_j : j \in D_{n,k} \text{ and } f(j) + 1 \neq n\},$$

making no modification to the action of  $\beta(g_n)$  on those  $u_j$  for which  $f(j) + 1 = n$ . We will call these cycles the top cycles of  $\beta(g_n)$ . We have the following picture of  $\beta$ , where  $n = f(3) + 1 = 6$  and we consider the interval  $D_{6,0}$ .



Finally  $\beta$  is defined trivially for all other points. Clearly  $\beta$  acts transitively. Write for  $m > 0$ ,  $\left(\bigcup_{k=0}^{m-1} W_k\right) \cup \{u_0, \dots, u_{m-1}\} = T_m$  and for  $m \geq 0$ ,  $T_{m!} = F_m$ . Thus  $F_m$  is invariant under  $\beta(G_m)$ . For each  $m$ , define a measure  $\theta_m$  on  $\text{Sub}(G)$  by letting  $\theta_m$  be the pushforward of the uniform measure on  $F_m$  under the map  $v \mapsto \text{stab}_\beta(v)$ . Let  $\theta$  be a weak-star limit point of  $\theta_m$ . By Lemma 2.1,  $\theta$  is an invariant random subgroup of  $G$ .

We claim that  $\theta$  is supported on  $\mathcal{M}_G$ . Let  $(Q_k)_{k=0}^\infty$  be an increasing sequence of finite subsets of  $G$  with  $\bigcup_{k=0}^\infty Q_k = G$ . For  $H \leq G$ , let  $Q_k/H = \{gH : g \in Q_k\}$ . Write, for  $n \geq 1, k \in \mathbb{N}$ ,

$$A_{n,k} = \{H \leq G : \alpha_n \text{ appears in } \tau_{G/H} \text{ within } Q_k/H\}.$$

By definition, if  $H \in \bigcup_{k=0}^\infty A_{n,k}$ , then  $\alpha_n$  appears in  $\tau_{G/H}$ . Therefore by Lemma 2.2, we have

$$\bigcap_{n=1}^\infty \bigcup_{k=0}^\infty A_{n,k} \subseteq \mathcal{M}_G.$$

Thus it suffices to show that for each  $n \geq 1$  we have  $\sup_{k < \infty} \theta(A_{n,k}) = 1$ . Fix  $n$  and  $\epsilon > 0$ . Since the set  $A_{n,k}$  is clopen for each  $k$ , it is enough to show the following:

**Claim 2.1.** *There is some  $k \in \mathbb{N}$ , such that for all  $m > 0$ , we have  $\theta_m(A_{n,k}) > 1 - \epsilon$ .*

Let  $t$  be large enough that Lemma 2.3(ii) holds for our chosen  $\epsilon$ . We now define five finite subsets of  $G$ .

- Let  $S_1 \subseteq G$  consist of  $\{1_G\}$  together with every word in the generators  $g_0, \dots, g_t$  with length at most  $\max_{1 \leq j \leq t} |V_j|$ . If  $f(j) \leq t$ , this choice will allow us to pass between points in  $W_j$  using an element of  $S_1$ .
- Let  $S_2 = \{1_G, g_0, \dots, g_{t+1}\}$ . If  $f(j) \leq t$ , this choice will allow us to pass to  $u_j$  from some point in  $W_j$  using an element of  $S_2$ .
- Let  $S_3$  consist of all words in the generators  $g_K, g_{2K}, g_{3K}$  of length at most  $3K$ , where  $K = K_n$  is the number provided by Lemma 2.3(i) for our fixed  $n$ . We will explain this choice later.
- Let  $S_4 = \{g_{n+1}\}$ . If  $f(l) = n$ , we will use  $g_{n+1}$  to pass from  $u_l$  to some point in  $W_l$ .
- Let  $S_5$  consists of all words in the generators  $g_1, \dots, g_n$  of length at most  $|V_n|$ . If  $f(l) = n$ , this choice will allow us to pass between any two points of  $W_l$  using an element of  $S_5$ .

Let  $k$  be large enough that  $Q_k$  contains  $S_5 \cdot S_4 \cdot S_3 \cdot S_2 \cdot S_1$ . We assert that the following implies Claim 2.1.

**Claim 2.2.** *If  $v \in W_j \cup \{u_j\}$  and  $f(j) \leq t$ , then  $\alpha_n$  appears in  $\tau_{G/\text{stab}_\beta(v)}$  within  $Q_k/\text{stab}_\beta(v)$ .*

Indeed, suppose Claim 2.2 holds and let  $m > 0$ . Note that  $C_{m!}(f)$  defined as in Lemma 2.3 is exactly  $|T_{m!}|$ . Thus we have

$$\theta_m(A_{n,k}) = \frac{1}{|T_{m!}|} \cdot |\{v \in T_{m!} : \text{stab}_\beta(v) \in A_{n,k}\}| \quad (2.1)$$

$$\geq \frac{1}{|T_{m!}|} \cdot |\{v \in T_{m!} : v \in W_j \cup \{u_j\} \text{ and } f(j) \leq t\}| \quad (2.2)$$

$$= \frac{1}{|T_{m!}|} \sum_{n=1}^t (|V_n| + 1) \cdot |\{j \in \{0, \dots, m! - 1\} : f(j) = n\}| \quad (2.3)$$

$$> 1 - \epsilon, \quad (2.4)$$

where

- (2.1) follows from the definition of  $\theta_m$ ,
- (2.2) follows from (2.1) by Claim 2.2,
- (2.3) follows from (2.2) since  $|W_j| = |V_{f(j)}|$ ,
- (2.4) follows from (2.3) by Lemma 2.1(ii).

Thus it remains to establish Claim 2.2.

Fix  $j$  with  $f(j) \leq t$ . By our choice of  $K$ , there is some  $l$  such that  $\lfloor j/K \rfloor = \lfloor l/K \rfloor$  and  $f(l) = n$ . Fix  $v \in W_j \cup \{u_j\}$ . Write  $H = \text{stab}_\beta(v)$  and let  $P = \{gH : \beta(g) \cdot v \in W_l\}$ . Since  $\beta(G_n)$  acts on  $W_l$  according to  $\alpha_n$ , it follows that  $\alpha_n$  appears in  $\tau_{G/H}$  within  $P$ . Therefore it is enough to show that  $P \subseteq Q_k/H$ , or equivalently  $W_l \subseteq \beta(Q_k) \cdot v$ . The idea is that we have chosen  $k$  large enough that we can reach any point in  $W_l$  from  $v$  using the  $\beta$  action of a word from  $Q_k$ .

By our choice of  $S_1$ , if  $v \in W_j$  there is an element  $\gamma \in S_1$  such that  $\beta(\gamma) \cdot v = w_j$  where  $w_j$  is the point in  $W_j$  connected to  $u_j$ . The connection between  $w_j$  and  $u_j$  is made by  $\beta(g_{f(j)+1})$ . We have  $g_{f(j)+1} \in S_2$  since  $f(j) \leq t$ . Thus  $u_j = \beta(\gamma) \cdot v$ , where  $\gamma \in S_2 \cdot S_1$ .

Note that our assumption on  $l$  guarantees that  $l$  lies between the same pair of multiples of  $K$  as  $j$  does. We would like to say that this allows us to pass from  $u_j$  to  $u_l$  using  $\beta(g_K)^i$  for some  $i \in [-K, K]$ . However, there is

the minor issue of the points  $u_d$  which are skipped the top cycles of  $\beta(g_K)$ . We can easily overcome this obstacle by noting that for any  $d$ , at most one of  $\beta(g_K), \beta(g_{2K})$  and  $\beta(g_{3K})$  skips over  $u_d$ , and therefore there is a word  $\gamma'$  in  $g_K, g_{2K}, g_{3K}$  of length at most  $3K$  such that  $\beta(\gamma') \cdot u_j = u_l$ . We have  $\gamma' \in S_3$ .

Since  $f(l) = n$ , we see that  $u_l$  is connected to  $W_l$  by  $\beta(g_{f(l)+1}) = \beta(g_{n+1})$ . Therefore  $\beta(g_{n+1}\gamma') \cdot v \in W_l$ . Since  $W_l \subseteq \beta(S_5) \cdot \beta(g_{n+1}\gamma') \cdot v$ , we have that  $W_l \subseteq \beta(Q_k) \cdot v$  and we are done.

### 2.3 Proofs of lemmas

*Proof of Lemma 2.1.* Let  $h_1, \dots, h_l, k_1, \dots, k_{l'}, g \in G$  and let  $\epsilon > 0$ . Let  $m$  be large enough that  $h_1, \dots, h_l, k_1, \dots, k_{l'}, g$  are words in the generators  $\{g_0, \dots, g_m\}$ . Write

$$C = \{H \leq G : h_1, \dots, h_l \in H \text{ and } k_1, \dots, k_{l'} \notin H\}.$$

Note that  $C$  is a clopen set and therefore there is some  $n \geq m$  such that

$$\theta(C) \approx_\epsilon \theta_n(C) \text{ and } \theta(gCg^{-1}) \approx_\epsilon \theta_n(gCg^{-1}). \quad (2.5)$$

Noting that  $F_n$  is  $\alpha(\langle g, h_1, \dots, h_l, k_1, \dots, k_{l'} \rangle)$  invariant we have

$$\begin{aligned} \theta_n(gCg^{-1}) &= \frac{1}{|F_n|} \cdot |\{v \in F_n : \alpha(gh_jg^{-1}) \cdot v = v \text{ for all } j \in \{1, \dots, l\} \\ &\quad \text{and } \alpha(gk_jg^{-1}) \cdot v \neq v \text{ for all } j \in \{1, \dots, l'\}\}| \\ &= \frac{1}{|F_n|} \cdot |\{v \in F_n : \alpha(h_j)\alpha(g^{-1}) \cdot v = \alpha(g^{-1}) \cdot v \text{ for all } j \in \{1, \dots, l\} \\ &\quad \text{and } \alpha(k_j)\alpha(g^{-1}) \cdot v \neq \alpha(g^{-1}) \cdot v \text{ for all } j \in \{1, \dots, l'\}\}| \\ &= \frac{1}{|F_n|} \cdot |\{w \in F_n : \alpha(h_j) \cdot w = w \text{ for all } j \in \{1, \dots, l\} \\ &\quad \text{and } \alpha(k_j) \cdot w \neq w \text{ for all } j \in \{1, \dots, l'\}\}| \\ &= \theta_n(C) \end{aligned}$$

Then from (2.5) we have  $\theta(C) \approx_{2\epsilon} \theta(gCg^{-1})$ .  $\square$

*Proof of Lemma 2.2.* It is clearly enough to find such  $V_n, \alpha_n$  such that for any  $\beta$  as in that lemma,  $\mathbf{s}_\beta$  is action-maximal and another sequence, also denoted below by  $V_n, \alpha_n$ , such that for any  $\beta$  as in that lemma,  $\lambda_\beta$  is representation-maximal. Then by interlacing these two sequences, we have a sequence that achieves both goals.



*Case 1:* We first find the sequence for which the appropriate  $\mathbf{s}_\beta$  is action-maximal. By [K1, Theorem 5.1], there is a countably infinite set  $J$  and a transitive action  $\alpha$  of  $G$  on  $J$  such that  $\mathbf{s}_\alpha$  is action-maximal. Identify  $(X, \mu)$  with  $2^J$  carrying the usual product measure. For a finite set  $T \subseteq J$  and  $\rho \in 2^T$ , write

$$N_\rho = \{x \in 2^J : x(v) = \rho(v) \text{ for all } v \in T\}.$$

For  $n \geq 1$ ,  $\epsilon > 0$  and a finite set  $T \subseteq J$ , let  $U_{n,\epsilon,T}$  be the set of all  $\mathbf{c} \in A(G, X, \mu)$  such that

$$\mu(\mathbf{s}_\alpha(g_k) \cdot N_\rho \cap N_\sigma) \approx_\epsilon \mu(\mathbf{c}(g_k) \cdot N_\rho \cap N_\sigma), \forall \sigma, \rho \in 2^T, k \in \{0, \dots, n-1\}.$$

Observe that the collection of all  $U_{n,\epsilon,T}$  is a neighborhood basis at  $\mathbf{s}_\alpha \in A(G, X, \mu)$ . Let  $(T_n)_{n=1}^\infty$  be an increasing sequence of finite subsets of  $J$  with  $\bigcup_{n=1}^\infty T_n = J$ . Write  $U_n = U_{n,2^{-n-|T_n|},T_n}$ . Then the sets  $U_n$  form a neighborhood basis at  $\mathbf{s}_\alpha$ . Note that for each  $n \geq 1$  and each  $k \in \{0, \dots, n-1\}$ , we can extend  $\alpha(g_k) \upharpoonright \left(T_n \cup \bigcup_{j=0}^{n-1} \alpha(g_j) \cdot T_n\right)$  to a permutation of  $J$

which is trivial on the complement of a finite set containing  $T_n \cup \bigcup_{j=0}^{n-1} \alpha(g_j) \cdot T_n$ .

Hence for each  $n \geq 1$ , we can find an action  $\hat{\alpha}_n$  of  $G$  on  $J$  with the following properties:

- (I)  $\hat{\alpha}_n(g_k) \cdot v = \alpha(g_k) \cdot v$ , if  $k \in \{0, \dots, n-1\}$  and  $v \in T_n$ .
- (II)  $\hat{\alpha}_n(g_k)$  acts trivially if  $k > n$ .
- (III) There is a  $\hat{\alpha}_n$ -invariant finite set  $V_n \subseteq J$  such that  $\hat{\alpha}_n \upharpoonright (J \setminus V_n)$  is trivial and  $\hat{\alpha}_n \upharpoonright V_n$  is transitive.

By (I) we see that  $\mathbf{s}_{\hat{\alpha}_n}(g_k) \cdot N_\rho = \mathbf{s}_\alpha(g_k) \cdot N_\rho$  for all  $\rho \in 2^{T_n}$  and  $k \in \{0, \dots, n-1\}$ . Therefore  $\mathbf{s}_{\hat{\alpha}_n} \in U_n$ . Write  $\alpha_n = \hat{\alpha}_n \upharpoonright V_n$ . By (II) all  $g_k, k > n$ , act trivially in  $\alpha_n$ . Observe that (III) implies that  $\mathbf{s}_{\hat{\alpha}_n} \cong \mathbf{s}_{\alpha_n} \times \iota$ , where  $\iota$  is the trivial action of  $G$  on a nonatomic standard probability space. Thus for each  $n \geq 1$  there is an isomorphic copy of  $\mathbf{s}_{\alpha_n} \times \iota$  in  $U_n$ .

Suppose  $\beta$  is a transitive action of  $G$  on a countably infinite set such that  $\alpha_n$  appears in  $\beta$  for each  $n \geq 1$ . Note that  $\mathbf{s}_\beta$  is ergodic (see, e.g., [KT, 2.1]). Then  $\mathbf{s}_{\alpha_n \upharpoonright G_n}$  is a factor of  $\mathbf{s}_{\beta \upharpoonright G_n}$  and hence  $\mathbf{s}_{\alpha_n \upharpoonright G_n} \times (\iota \upharpoonright G_n)$  is a factor of

$\mathfrak{s}_\beta \upharpoonright_{G_n} \times (\iota \upharpoonright_{G_n})$ . Using the fact that the definition of  $U_n$  depends only on  $G_n$ , this implies that for each  $n \geq 1$  there is an isomorphic copy of  $\mathfrak{s}_\beta \times \iota$  in  $U_n$ . Therefore there is a sequence of isomorphic copies of  $\mathfrak{s}_\beta \times \iota$  in  $A(G, X, \mu)$  which converges to  $\mathfrak{s}_\alpha$ . Since the isomorphic copies of  $\mathfrak{s}_\alpha$  are dense in  $A(G, X, \mu)$ , this implies that the isomorphic copies of  $\mathfrak{s}_\beta \times \iota$  are dense in  $A(G, X, \mu)$ .

By [T, Theorem 3.11], we see that any ergodic action  $\mathbf{d}$  of  $G$  is weakly contained in almost every ergodic component of  $\mathfrak{s}_\beta \times \iota$ . In particular, any ergodic action  $\mathbf{d}$  of  $G$  is weakly contained in  $\mathfrak{s}_\beta$  and therefore the isomorphic copies of  $\mathfrak{s}_\beta$  are dense in the ergodic actions. Since  $G$  does not have Property (T), [K, Theorem 12.2] implies that the isomorphic copies of  $\mathfrak{s}_\beta$  are dense in  $A(G, X, \mu)$ .

*Case 2:* We next find a sequence  $V_n, \alpha_n$ , for which the appropriate  $\lambda_\beta$  is representation-maximal. We start with a transitive action  $\alpha$  of  $G$  on a countably infinite set  $J$  such that  $\lambda_\alpha$  is representation-maximal (see [K1, Theorem 5.5]). Then proceed as in the proof of Case 1 to find  $V_n, \alpha_n$  such that for some isomorphic copy  $\sigma_n$  of  $\lambda_{\alpha_n} \oplus \infty 1_G$ ,  $(\sigma_n)$  converges to  $\lambda_\alpha$ , where  $1_G$  is the trivial one-dimensional representation of  $G$  and  $\infty 1_G$  the direct sum of countably many copies of  $1_G$ , i.e., the trivial representation on a separable, infinite-dimensional Hilbert space. Let now  $\beta$  be as above. Then the isomorphic copies of  $\lambda_\beta \oplus \infty 1_G$  converge to  $\lambda_\alpha$ . By a result of Hjorth, see [K, H.7], the irreducible representations are dense in  $\text{Rep}(G, H)$ . Every irreducible representation  $\pi$  is  $\preceq \lambda_\alpha$  and thus  $\preceq_Z \lambda_\alpha \preceq_Z \lambda_\beta \oplus \infty 1_G$ , where  $\preceq_Z$  is weak containment in the sense of Zimmer. Recall that  $\sigma \preceq_Z \rho$  iff  $\sigma$  is in the closure of the isomorphic copies of  $\rho$ . Also  $\sigma \preceq_Z \rho \implies \sigma \preceq \rho$  and for  $\sigma$  irreducible,  $\sigma \preceq_Z \rho \iff \sigma \preceq \rho$  (see [BHV, page 397] and [K, page 209]). Then by [AE, Proposition 3.5]  $\pi$  is a subrepresentation of an ultrapower of  $\lambda_\beta \oplus \infty 1_G$ , which is of course of the form  $\lambda_\beta^* \oplus \eta^*$ , where  $\lambda^*$  is an ultrapower of  $\lambda_\beta$  and  $\eta^*$  a trivial representation of  $G$  on a Hilbert space  $H^*$ . Let  $H_1$  be the space on which this subrepresentation acts, which is a  $G$ -invariant subspace of the direct sum of the space of  $\lambda_\beta^*$  and  $H^*$ . Then if  $v \in H^*$  and  $v_1$  is its projection on  $H_1$ ,  $v_1$  is  $G$ -invariant, so as  $\pi$  is irreducible,  $v_1 = 0$ , i.e.,  $H^* \perp H_1$ . Thus  $H_1$  is contained in the space of  $\lambda_\beta^*$ , i.e.,  $\pi$  is a subrepresentation of  $\lambda_\beta^*$ , so  $\pi \preceq_Z \lambda_\beta$ . Thus the isomorphic copies of  $\lambda_\beta$  are dense in  $\text{Rep}(G, H)$ , i.e.,  $\lambda_\beta$  is representation-maximal.  $\square$

*Proof of Lemma 2.3.* Note that letting for  $n \geq 1$ ,  $A_n = f^{-1}(\{n\})$  the statement of the lemma is equivalent to the existence of a partition  $\mathbb{N} = \bigsqcup_{n \geq 1} A_n$

with the following properties:

(i) For each  $n \geq 1$  there is positive integer  $K_n$  such that  $A_n$  intersects each interval  $I_i^n = [iK_n, (i+1)K_n)$ ,  $i = 0, 1, 2, \dots$ .

(ii) Let  $g : \mathbb{N}^+ \rightarrow \mathbb{N}^+$  be defined by  $g(n) = |V_n| + 1$ , where  $V_n$  is as in Lemma 2.2. Then we have that for each  $\epsilon > 0$ , there is  $t > 0$ , such that for all  $m > 0$ :

$$\frac{\sum_{n>t} (|A_n \cap m| \cdot g(n))}{\sum_n (|A_n \cap m| \cdot g(n))} < \epsilon,$$

where we identify here  $m$  with  $\{0, 1, \dots, m-1\}$ .

To construct  $A_n, K_n$ , first chose  $a_2 < a_3 < \dots$  to be large enough so that  $a_n$  is divisible by 3 and

$$\sum_{n=2}^{\infty} \frac{1}{a_2 \cdots a_n} < \frac{1}{3} \text{ and } \frac{a_n}{3} > \frac{g(n)2^n}{g(n-1)}.$$

We let  $A'_1 = \{2i : i \in \mathbb{N}\}$  and also put  $K_1 = 2, K_n = 2a_2 \cdots a_n$  for  $n \geq 2$ . We will then inductively define pairwise disjoint  $A_2, A_3, \dots$ , which are also disjoint from  $A'_1$ , to satisfy (ii) above and so that for  $n \geq 2$ ,  $A_n$  has exactly one member in each interval  $I_i^n$  as above, and finally we put  $A_1 = \mathbb{N} \setminus \bigcup_{n=2}^{\infty} A_n$ .

So assume that  $A'_1, A_2, \dots, A_{n-1}$  have been constructed (this is just  $A'_1$ , if  $n = 2$ ). To find  $A_n$ , so that (i) above is satisfied, it is enough to have for each  $i$ ,

$$K_n > \frac{3}{2} |(A'_1 \cup A_2 \cup \cdots \cup A_{n-1}) \cap I_i^n|.$$

But

$$|(A'_1 \cup A_2 \cup \cdots \cup A_{n-1}) \cap I_i^n| = a_2 \cdots a_n + a_3 \cdots a_n + \cdots + a_{n-1}a_n + a_n,$$

so this follows from  $\sum_{n=2}^{\infty} \frac{1}{a_2 \cdots a_n} < \frac{1}{3}$ . Also for  $i = 0$ , we can choose the element of  $A_n$  in  $[0, K_n)$  to be  $\geq \frac{K_n}{3}$ .

We finally check that (ii) is satisfied. Fix  $\epsilon > 0$  and choose  $t > 1$  so that  $\sum_{n>t} 2^{-n} < \epsilon$ . Consider now any  $m > 0$  and  $n > t$ .

*Case 1.*  $m \geq K_n$ . Then for some  $s > 1$ , we have that  $m \in I_{s-1}^n$  and  $|A_n \cap m| \leq s$ , while

$$\sum_n |A_n \cap m| \cdot g(n) \geq |A_{n-1} \cap m| \cdot g(n-1) \geq (s-1)a_n \cdot g(n-1)$$

so

$$\frac{|A_n \cap m| \cdot g(n)}{\sum_n |A_n \cap m| \cdot g(n)} \leq \frac{s \cdot g(n)}{(s-1) \cdot g(n-1)} \cdot \frac{1}{a_n} < 2^{-n}.$$

*Case 2.*  $m < K_n$ . Then either  $m \leq \frac{K_n}{3}$  and  $|A_n \cap m| = 0$  or  $m > \frac{K_n}{3}$  and  $|A_n \cap m| \leq 1$ , in which case also

$$|A_{n-1} \cap m| \geq \frac{a_n}{3}.$$

So for any  $m < K_n$ ,

$$\frac{|A_n \cap m| \cdot g(n)}{\sum_n |A_n \cap m| \cdot g(n)} \leq \frac{g(n)}{\left(\frac{a_n}{3}\right)g(n-1)} < 2^{-n}.$$

Thus for any  $n > t$ , we have

$$\frac{|A_n \cap m| \cdot g(n)}{\sum_n |A_n \cap m| \cdot g(n)} < 2^{-n}$$

and so

$$\frac{\sum_{n>t} (|A_n \cap m| \cdot g(n))}{\sum_n (|A_n \cap m| \cdot g(n))} < \epsilon$$

□

### 3 Proof of Theorem 1.2

We note that if  $\lambda_{G/H}$  is representation-maximal, then  $H$  is not amenable. This is because  $1_G \preceq \lambda_{G/H}$  implies  $\tau_{G/H}$  is amenable (see [KT, Theorem 1.1]).

We will use the notion of a random Bernoulli shift over an invariant random subgroup; we refer the reader to [T, Section 5.3] and [AGV, Proposition 45] for details. Let  $\theta$  be the invariant random subgroup constructed in Theorem 1.1 and let  $\mathbf{s}_\theta$  be the  $\theta$ -random Bernoulli shift. Note that for almost every ergodic component  $\mathbf{b}$  of  $\mathbf{s}_\theta$ , almost all stabilizers of  $\mathbf{b}$  lie in  $\mathcal{M}_G$  and hence the type of  $\mathbf{b}$  is supported on  $\mathcal{M}_G$ . Fix such an action  $\mathbf{b}$ . Let  $(Y, \nu)$  be the underlying space of  $\mathbf{b}$ .

For  $y \in Y$  write  $H_y = \text{stab}_{\mathbf{b}}(y)$ . By [DG, Proposition 8] we have  $\lambda_{G/H_y} \preceq \kappa_0^{\mathbf{b}}$  for  $\nu$ -almost every  $y \in Y$ . Since the type of  $\mathbf{b}$  is supported on  $\mathcal{M}_G$ , for  $\nu$ -almost every  $y$  we have that  $\lambda_{G/H_y}$  is representation-maximal and so  $\kappa_0^{\mathbf{b}}$  is representation-maximal. Let  $\mathbf{a} = \mathbf{b} \times \mathbf{s}_G$ . Then  $\mathbf{a}$  is free and ergodic and

$\kappa_0^{\mathbf{a}}$  is representation-maximal. Suppose, toward a contradiction, that  $\mathbf{a}$  were action-maximal.

Let  $S \subseteq G^2$  be the collection of all pairs  $(g, h)$  such that  $\langle g, h \rangle$  is nonamenable. Since  $\lambda_{G/H_y}$  is representation-maximal for  $\nu$ -almost every  $y \in Y$ , and so  $H_y$  is not amenable, we see that  $S \cap H_y^2$  is nonempty for  $\nu$ -almost every  $y$ . Let  $\phi : \mathbb{N} \rightarrow S$  be an enumeration of  $S$ . For  $y \in Y$  let  $\phi_y = \min\{n : \phi(n) \in H_y^2\}$ . Then there is some  $k \in \mathbb{N}$  such that  $\nu(\{y : \phi_y = k\}) > 0$ . Write  $A = \{y : \phi_y = k\}$  and let  $N$  be the subgroup of  $G$  generated by the coordinates of  $\phi(k)$ . Note that for  $y \in A$ , we have  $N \subseteq H_y$ , and so  $\mathbf{b} \upharpoonright N$  is trivial on  $A$ . By [K, Page 74], since  $\mathbf{a}$  is action-maximal for  $G$ , we have that  $\mathbf{a} \upharpoonright N$  is action-maximal for  $N$ . Observe that

$$\mathbf{a} \upharpoonright N = (\mathbf{b} \upharpoonright N) \times (\mathbf{s}_G \upharpoonright N) \cong (\mathbf{b} \upharpoonright N) \times (\mathbf{s}_N)^{\mathbb{N}} \cong (\mathbf{b} \upharpoonright N) \times \mathbf{s}_N.$$

So writing  $\mathbf{c} = (\mathbf{b} \upharpoonright N) \times \mathbf{s}_N$ , we have that  $\mathbf{c}$  is action-maximal for  $N$ .

By [T, Theorem 3.11], this implies that any ergodic action  $\mathbf{d}$  of  $N$  is weakly contained in almost every ergodic component of  $\mathbf{c}$ . Note that if  $y \in A$ , then  $\iota_{\{y\}} \times \mathbf{s}_N \cong \mathbf{s}_N$  is an ergodic component of  $\mathbf{c}$ , where by  $\iota_{\{y\}}$  we mean the trivial action of  $N$  on the one-point space  $\{y\}$ . Therefore  $\mathbf{d} \preceq \mathbf{s}_N$ . Since  $N$  does not have property (T), the ergodic actions of  $N$  are dense in  $A(N, X, \mu)$  (see [K, 12.2]), so the isomorphic copies of  $\mathbf{s}_N$  are dense in  $A(N, X, \mu)$ . But by [K, Proposition 13.2] this contradicts the fact that  $N$  is nonamenable.

**Remark 3.1.** For  $G = \mathbf{F}_\infty$ , let  $\mathbf{a}$  be as in Theorem 1.2. Then for any irreducible  $\pi$  we have  $\pi \preceq \kappa_0^{\mathbf{a}}$ , so  $\pi \preceq_Z \kappa_0^{\mathbf{a}}$ . Thus, as the irreducible representations are dense,  $\pi \preceq_Z \kappa_0^{\mathbf{a}}$ , for all  $\pi$ . Thus there is a free ergodic action  $\mathbf{b}$  such that  $\kappa_0^{\mathbf{b}} \preceq_Z \kappa_0^{\mathbf{a}}$  but  $\mathbf{b} \not\preceq \mathbf{a}$ , which is a somewhat stronger negative answer to Problem 1.1.

## References

- [AE] M. Abért and G. Elek, The space of actions, partition metric and combinatorial rigidity, *arXiv:1108.2147*.
- [AGV] M. Abért, Y. Glasner, and B. Virág, Kesten’s theorem for invariant random subgroups, *Duke Math. J.*, **163(3)** (2014), 466–488.
- [BHV] B. Bachir, P. de la Harpe and A. Valette, *Kazhdan’s Property (T)*, Cambridge Univ. Press, 2008.

- [CK] C.T. Conley and A.S. Kechris, Measurable chromatic and independence numbers for ergodic graphs and group actions, *Groups Geom. Dyn.*, **7** (2013), 127–180.
- [DG] A. Dudko and R. Grigorchuk. On spectra of Koopman, groupoid and quasi-regular representations. *arXiv:1510.00897v2*.
- [K] A. S. Kechris. *Global aspects of ergodic group actions*, Amer. Math. Society, 2010.
- [K1] A. S. Kechris. Weak containment in the space of actions of a free group, *Israel J. Math.*, **189** (2012), 461–507.
- [KT] A. S. Kechris and T. Tsankov. Amenable actions and almost invariant sets, *Proc. Amer. Math. Soc.*, **136(2)** (2008), 687–697.
- [T] R. D. Tucker-Drob. Weak equivalence and non-classifiability of measure preserving actions, *Ergodic Theory Dynam. Systems*, **35(1)** (2015), 293–336.

Department of Mathematics  
California Institute of Technology  
Pasadena, CA 91125

pjburton@caltech.edu, kechris@caltech.edu