ARE THERE CHAOTIC TILINGS?

by

Daniel Berend[1,2] * and Charles Radin[1] **

1 Mathematics Department, University of Texas, Austin, TX 78712

2 Department of Mathematics and Computer Science, Ben-Gurion University, Beer-Sheva 84105, Israel

Abstract

We develop a class of examples in the form of tiling dynamical systems for use as toy models in statistical mechanics, to analyze the possible existence of disordered crystals. We give the first such models which are disordered in the sense of having no discrete spectrum.

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1. Introduction

Ten years ago, David Ruelle published the paper "Do turbulent crystals exist?" [16], in which he suggested the existence of real materials which in thermal equilibrium at low temperature would be quite different microscopically from the usual periodic crystals; the suggested difference would be demonstrated by a diffraction spectrum which was absolutely continuous, even at zero temperature.

Ruelle's argument was based on a comparison of the usual classical statistical mechanical formalism with a typical dynamical system with $\mathbf{R}^3$ action ($\mathbf{R}^3$ representing spatial translations), but without any detailed consideration of the structural role played by interacting particles in the former.

The present paper is motivated by the same problem, but with a different premise. We have chosen to concentrate on the special features which may be due to the role played by the interacting particles in statistical mechanics, with the aim to determine the qualitative features of (the ground states of) generic classical statistical mechanical models with short range interactions. It is well known [16] that no such model has ever been proven to exhibit an ordered (crystalline) phase; presumably the reason is the difficulty in analyzing such models. To obtain results we first restrict attention to zero temperature, and then we distort the models to that of tiling dynamical systems (defined below), as is sometimes done in analyzing quasicrystals [18]. (Roughly speaking, in a tiling dynamical system the phase space consists of the tilings of Euclidean n-space by copies of some finite set of shapes called tiles; intuitively, the way in which neighboring pairs of tiles need to fit together in a tiling replaces the short range interaction of mechanics.) It has been proven [8] that, generically, statistical mechanical models are uniquely ergodic with respect to spatial translations at zero temperature, and so we use this as an assumption in our models. In summary, the problem of the qualitative behavior of low temperature matter is here translated into: **What is the range of qualitative behavior of uniquely ergodic tiling dynamical systems; in particular, do there exist such systems with absolutely continuous spectrum?**

We introduce now some definitions. A *tiling* of Euclidean n-space, E^n, is a decomposition of E^n into a union of "tiles" where:

(a) there is a fixed finite set $\mathcal{P}$ of "prototiles", which are homeomorphic images of the closed n-ball;
(b) each tile is an isometric copy of some prototile,
(c) the interiors of the tiles do not overlap,
(d) the isometries in (b) are restricted to some fixed subgroup G of the full isometry group of E^n.

One of our subsidiary goals is to unify the work done on two types of tilings of E^n: the so-called "Robinson-like tilings" and the "Penrose-like tilings", the chief difference being that in the former only discrete (lattice) translations are used (to make tiles from prototiles), whereas in the latter the full translation group of E^n is used [5,8,17,18]. (A significant motivation is the connection with quasicrystals [8,17,18].)

Before discussing tiling any further, we need to endow the space $V(\mathcal{P})$ (assumed nonempty) of all tilings by some given set $\mathcal{P}$ of prototiles with a topology. The motivating idea for the topology is that tilings should be close if they differ only slightly inside some large bounded region. To implement this we use the Hausdorff distance of a pair of compact subsets A, B of E^n, defined as $h[A,B] = \max\{s(A,B), s(B,A)\}$, where $s(A,B) = \sup_{a\in A}\inf_{b\in B}\|a-b\|$. A finite set of nonoverlapping tiles will be called a *swatch*. We define a countable base for the topology on $V(\mathcal{P})$, using some countable dense subset G' of the topological subgroup G (usually $\mathbf{Z}^n$ or $\mathbf{R}^n$) of the isometry group of E^n, as follows. Given a positive integer k, a set of positive rationals $\{r_j\}_1^k$, and a swatch of tiles $\{g_j'(P_j')\}_1^k$ (where $g_j' \in G'$, $P_j' \in \mathcal{P}$), we define the open set consisting of all tilings containing a swatch $\{g_j(P_j)\}_1^k$ such that $h[g_j(P_j), g_j'(P_j')] < r_j$ for all $j \le k$. We note that the space $V(\mathcal{P})$, of tilings from a given prototile set $\mathcal{P}$, is compact and metrizable, and G acts continuously on $V(\mathcal{P})$ [10].

As said earlier, our interest here is in the order/disorder properties of tilings. Now questions about order properties are best discussed in the framework of ergodic theory, since this allows one to make use of probabilistic notions of order through the spectral or mixing properties of certain dynamical systems associated with tilings of E^n. By a *dynamical system* we mean a (compact metrizable) space X on which some topological group H is continuously represented by homeomorphisms. In most classical examples H is $\mathbf{Z}$ or $\mathbf{R}$ but we will also need $\mathbf{Z}^n$ and $\mathbf{R}^n$. Dynamical systems with $H = \mathbf{Z}^n$ will be called *discrete*, those with $\mathbf{R}^n$ *continuous*. If there is also a probability measure μ on X, invariant under the group action, we have further techniques going under the heading of ergodic theory. We will be couching much of our discussion in the framework of ergodic theory. For example, we mentioned above that one of our primary objectives is to unify the work done on two types of tilings of E^n; this will be accomplished by associating similar dynamical systems with such tilings, the main difference being that for the former type one has the action of $\mathbf{Z}^n$ and for the latter $\mathbf{R}^n$.

To illustrate the relationship between the two types of dynamical systems to be considered, we begin with the following (oversimplified) situation. Consider any subshift (X,T) over a finite alphabet $\mathcal{A}$; that is, $X \subseteq \mathcal{A}^{\mathbf{Z}}$ is compact and T is lattice translation on X in the usual sense that $T : x = (x_j)_{j\in\mathbf{Z}} \in X \longrightarrow Tx \in X$, where $(Tx)_j = x_{j+1}$. (We will need to refer on occasion to the cylinder sets $\mathcal{C}_a \equiv \{x \in X \,:\, x_0 = a\}$, $a \in \mathcal{A}$.) Then, given a positive real-valued function f on the alphabet $\mathcal{A}$, we associate with this subshift (X,T), which is a **discrete** dynamical system, the group $\mathbf{Z}$ being represented by powers of T, the **continuous** dynamical system (X_f, T_f) defined as follows. X_f is the subset of all tilings of $\mathbf{R}$ by translations of closed intervals $[0,a]$, where a is in the range of f, and the sequence of intervals I, of length $|I|$, is such that any corresponding sequence of letters $f^{-1}(|I|) \in \mathcal{A}$ is in X. X_f is easily seen to be a closed subset of the space of all tilings by such intervals, and invariant under translations, which are denoted by $\{T_f^t : t \in \mathbf{R}\}$. (An illustrative example has for X the orbit closure under translation of the "Morse sequence", and an arbitrary fixed f.)

By construction, the two spaces X_f and X are closely related, but of course the dynamical action is quite different for the two, and that for X_f depends in any case on the function f. In particular, any rational relation between the values of f (for example if f is constant) essentially introduces a relation in the elements of X_f between the roles that the letters of $\mathcal{A}$ play in the elements of X, so it is natural to focus attention on those f for which the range is a rationally independent set. We are primarily interested, for reasons discussed below, in cases where there are natural invariant probability measures μ_X (resp. μ_Y) on (X, T) (resp. (X_f, T_f)), and we want to know the relation between the spectra of the pair of related dynamical systems. To remind the reader of the definition of spectrum, consider the complex Hilbert subspace $\mathcal{H}_X \subset L^2(X, \mu_X)$ (resp. $\mathcal{H}_Y \subset L^2(X_f, \mu_Y)$) which is the orthogonal complement of the subspace of constant functions, and the unitary operators T^j on $\mathcal{H}_X$ for $j \in \mathbf{Z}^n$ (resp. T_f^t on $\mathcal{H}_Y$ for $t \in \mathbf{R}^n$), with spectral resolutions $T^j = \int_{[0,1]^n} \exp(2\pi i\lambda \cdot j)\, dE_\lambda$ (resp. $T_f^t = \int_{\mathbf{R}^n} \exp(2\pi i\lambda \cdot t)\, dE_\lambda$.) Such spectral resolutions define [12], for nonzero vectors ψ, measures $\mu_\psi = (\psi,\, dE_\lambda\psi)$ on $[0,1]^n$ (resp. $\mathbf{R}^n$). We are interested in the smoothness in λ of such measures as ψ varies over the unit sphere of $\mathcal{H}_X$ (resp. $\mathcal{H}_Y$). Every measure $(\psi,\, dE_\lambda\psi)$ decomposes uniquely into three parts: discrete, singular continuous and absolutely continuous [15]. Similarly, the space $\mathcal{H}_X$ (resp. $\mathcal{H}_Y$) admits a decomposition into orthogonal subspaces such that for a vector ψ in one of these, the measure μ_ψ is indecomposable (that is, has only one part) [11]. A dynamical system is said to have purely discrete (resp. singular continuous, resp. absolutely continuous) spectrum if exactly one of these subspaces is nonzero, and the measures corresponding to its vectors are discrete (resp. singular continuous, resp. absolutely continuous.)

We continue our discussion by specifying (X, T) as the substitution dynamical system determined by the substitution ξ:

$$\xi(0) = 0101, \quad \xi(1) = 1110 \ . \tag{1}$$

Thus (X, T) is the subshift defined as follows. For each finite sequence B of 0's and 1's let $\xi(B)$ be the finite sequence obtained by applying the above substitution rules to each 0 and 1 in B. Then define $D_0 = \{0, 1\}$, $D_j = \bigcup_{B\in D_{j-1}} \xi(B)$ for each $j \in \mathbf{N}$, and $D = \bigcup_{j\geq 0} D_j$. The "subblocks" of a finite or infinite sequence $x = (x_j)_{j\in S} \in \{0,1\}^S$, where S is a subinterval of $\mathbf{Z}$, are the restrictions of the "function" x to finite subintervals of S. Then, finally, X is defined as the subset of $\{0,1\}^{\mathbf{Z}}$ consisting of those x all of whose subblocks appears in elements of D. We note that (X, T) is uniquely ergodic [7] (i.e., there exists a unique T-invariant probability measure on X), which easily implies that (X_f, T_f) is also uniquely ergodic for any f.

2. Preliminary results

Our first result is

Lemma 1. Let (X,T) be the substitution dynamical system defined by the substitution ξ given by (1). If $f(0)$ and $f(1)$ are rationally independent, then the associated **continuous** system (X_f, T_f) has no discrete spectrum.

Before giving the proof, it is appropriate to note the similarity of this result to an elegant result of Dekking and Keane [3]. The definition given above, of the continuous system (X_f, T_f) associated with the discrete system (X,T), is equivalent to what is known in ergodic theory [1,2] as "the flow under $\tilde{f}$ over T", where $\tilde{f}$ is the continuous function on X given by $\tilde{f}(x) = f(x_0)$, and is defined as follows. The underlying compact space for the system is $X'_f \equiv \{(x,s) \in X \times \mathbf{R} : 0 \le s \le \tilde{f}(x)\}$ in which all pairs of points $(x, \tilde{f}(x))$ and $(Tx, 0)$ are identified, and the dynamics is determined by ${T'_f}^t(x,s) \equiv (x, s+t)$ for $0 \le t < \tilde{f}(x) - s$, given the identification in the definition of X'_f. This system (X'_f, T'_f) can be viewed as follows. Each point x in the "base" X generates a fiber $\{(x,s) : 0 \le s \le \tilde{f}(x)\}$ in X'_f of height $\tilde{f}(x)$, and the dynamics moves $(x,0)$ up the fiber at constant speed until it hits the "ceiling" at which it "jumps" to the point $(Tx,0)$ and continues the motion. Of course, the jump is actually continuous because of the identification.

There is an analogous construction, which however associates with (X,T) another **discrete** system, $(\hat{X}_f, \hat{T}_f)$. Here the function f on $\mathcal{A}$ assumes positive **integer** values. The system $(\hat{X}_f, \hat{T}_f)$, called a tower over (X,T), is defined as follows. $\hat{X}_f \equiv \{(x,j) \in X \times \mathbf{Z} : 0 \le j \le f(x_0)\}$ in which we identify pairs of points of the form $(x, f(x_0))$ and $(Tx, 0)$. The dynamics is determined by $\hat{T}_f(x,j) \equiv (x, j+1)$ whenever $0 \le j < f(x_0)$, given the identification in the definition of $\hat{X}_f$.

The above tower constuction is conceptually very similar to that of the flow under a function. And as was the case with the flow under a function, it is sometimes preferable to have the following alternative picture of this tower. It is easy to see that $(\hat{X}_f, \hat{T}_f)$ is isomorphic to the subshift defined as the sequences in $\mathcal{A}^{\mathbf{Z}}$ obtained from those in X by replacing each letter a by $f(a)$ copies of itself; in a sense, the tower "stretches" each letter $a \in \mathcal{A}$ by the factor $f(a)$. Given this, it is not so surprising that our proof of Lemma 1 is very similar to that by Dekking and Keane of

Lemma 2 [3]. Let (X,T) be the dynamical system defined by the substitution ξ given by (1). If $f(0) = 2$ and $f(1) = 1$, then the associated **discrete** system $(\hat{X}_f, \hat{T}_f)$ has no discrete spectrum.

At this point we give the proof of Lemma 1.

Proof. Suppose we have an eigenfunction g of the family T_f^t:

$$T_f^t g(x,s) = e^{2\pi i \lambda t} g(x,s) \ . \tag{2}$$

Let u be the function on X defined as the restriction of g to some height s_0, $0 \leq s_0 < \min\{f(0), f(1)\}$ (where (2) holds for all t a.e.):

$$u(x) = g(x, s_0), \qquad x \in X \ .$$

In view of (2) we have:

$$Tu(x) = \begin{cases} e^{2\pi i\lambda f(0)}u(x), & x \in \mathcal{C}_0 \\ e^{2\pi i\lambda f(1)}u(x), & x \in \mathcal{C}_1 \ . \end{cases}$$

Defining a sequence of functions $f_n \, : \, X \to \mathbf{R}_+ \, , \; n \geq 1$, by

$$f_n(x) = \sum_{k=0}^{4^n-1} f([T^k x]_0) \ ,$$

we have:

$$T^{4^n} u \ = \ e^{2\pi i\lambda f_n} u \ .$$

As in the proof of Theorem 1 in [3], this implies that $e^{2\pi i\lambda f_n} \xrightarrow{L^2} 1$, or, equivalently,

$$\lambda f_n \xrightarrow{L^2} 0 \ (\text{mod } 1) \tag{3}$$

(where $\xrightarrow{L^2}$ denotes convergence in L^2 norm). Again as in [3] we can find sets $A_n \subseteq X$ whose measures are bounded away from 0 such that

$$f_n(x) \ = \ \frac{4^n-1}{3}f(0) + \frac{2\cdot 4^n+1}{3}f(1) \, , \qquad x \in A_n \ . \tag{4}$$

From (3) and (4) it follows that:

$$4^n\lambda\frac{f(0)+2f(1)}{3} \longrightarrow \lambda\frac{f(0)-f(1)}{3} \ (\text{mod } 1) \ . \tag{5}$$

In particular, the right hand side is invariant under multiplication by 4 modulo 1:

$$4\lambda\frac{f(0)-f(1)}{3} = \lambda\frac{f(0)-f(1)}{3} \ (\text{mod } 1) \ .$$

Consequently, $\lambda f(1) = \lambda f(0) + l$ for some $l \in \mathbf{Z}$. Again by (5), $3\lambda f(0) + 2l$ is a (2-adic) rational, so both $\lambda f(0)$ and $\lambda f(1)$ are rationals. As $f(0)$ and $f(1)$ are rationally independent, this implies $\lambda = 0$. This means in turn that u is a T-invariant function. As T is ergodic, u is constant a.e. Also, since $\lambda = 0$ the function g is constant as a function of s for each fixed x. Thus, g is constant in both variables. ∎

3. Tilings and Order

One could try to understand the order properties of all types of tilings [13], but we are primarily concerned with prototile sets which **force** interesting features in **all** their tilings. This line of development is due to Wang [19], who (motivated by certain questions in logic) initiated the study of sets of prototiles which can tile the plane but **only nonperiodically**. (A tiling is *periodic* if it consists of a lattice of translates of one of its swatches.) Two of the early notable examples are the sets of prototiles discovered by Robinson [14] and by Penrose [4] which can tile the plane but only nonperiodically.

The work of Wang eventually led to a succession of efforts to find sets of prototiles for which the tilings are of necessity (that is, for which **all** tilings are) more and more "disordered" – a notion which is inherently vague, but which we specify for this work below. This in turn has developed beyond tilings into a separate subfield of mathematics, in which one analyzes the order and symmetry properties which are determined by the optimization of a function of many weakly interacting variables [8].

A major step forward in this field was the development by Mozes [6] of a technique which combined the ideas in Robinson's example with the mathematics of substitution dynamical systems. Simply put, Mozes showed how, given any two substitution dynamical systems (satisfying some mild conditions), one can construct a (large) set of prototiles in E^2, each of which is a unit square with small bumps and dents on its edges, all centered at the origin in the plane and with edges aligned, such that the associated discrete dynamical system (with $\mathbf{Z}^2$ "dynamics") is uniquely ergodic, and is measure- theoretically isomorphic to the product of the two given substitution dynamical systems. If the systems are (X_1, T_1) and (X_2, T_2), with alphabets $\mathcal{A}_1$ and $\mathcal{A}_2$, the product is $(X_1 \times X_2, \mathbf{T})$, where $\mathbf{T} = (T_1^j \times T_2^k)_{j,k=-\infty}^{\infty}, T_1^j \times T_2^k : (x_1, x_2) \longrightarrow (T_1^j x_1, T_2^k x_2)$. The isomorphism is a simple one-to-one correspondence between the tilings and the elements of $X_1 \times X_2$; there is a many-to-one correspondence between the prototiles and pairs $(a, b) \in \mathcal{A}_1 \times \mathcal{A}_2$, and each tiling, which is a two-dimensional array of (essentially) unit square tiles, is naturally identifiable with the corresponding element of $X_1 \times X_2$.

This was used in [9] as follows. In proving Lemma 2, Dekking and Keane noted that the tower associated with (1) is again a substitution dynamical system. Let $(V(\mathcal{P}), \mathbf{Z}^n)$ be the tiling system obtained by Mozes' construction with both factors being the tower associated with (1).

Theorem 1 [9]. $(V(\mathcal{P}), \mathbf{Z}^n)$ is a **discrete**, uniquely ergodic dynamical system which is weakly mixing; that is, has no discrete spectrum.

We now modify the above to adapt it to continuous tilings. Begin by using the substitution dynamical system (X, T) defined by (1) for both factors in Mozes' construction. Next modify the shapes of the "square" prototiles that the construction produces, by "stretching" each one associated with the pair

$(a, b) \in \{0, 1\} \times \{0, 1\}$ to a rectangle with horizontal edges $f(a)$ and vertical edges $f(b)$, where $f(0)$ and $f(1)$ are fixed and rationally independent. Lemma 1 then implies that the continuous tiling system thus produced is weakly mixing. Let $(V(\mathcal{P}), \mathbf{R}^n)$ be this tiling system.

Theorem 2. $(V(\mathcal{P}), \mathbf{R}^n)$ is a **continuous**, uniquely ergodic dynamical system which is weakly mixing; that is, has no discrete spectrum.

4. Closing Remarks

Notice that for our examples there are invariant probability measures which we use to describe the disorder. It is essential for our purposes that there is no choice involved with these measures: they are uniquely defined by the prototile sets themselves. If one could **choose** an invariant measure, there would be no depth to the subject: one could very easily find a prototile set with associated dynamics which was very wild. For example, the one-dimensional continuous tiling system, with prototile set consisting of two intervals of incommensurate lengths, is strongly mixing if one chooses an appropriate measure; just use the method of the introduction, applying the flow under a function to the "Bernoulli" shift, namely $X = \{0, 1\}^{\mathbf{Z}}$, equipped with the product measure μ_B for which $\mu_B(\mathcal{C}_0) = \mu_B(\mathcal{C}_1) = 1/2$. (Physically, there is no surprise that a noninteracting particle system has a unique zero temperature Gibbs state which has absolutely continuous spectrum. The result is uninteresting because it does not represent a single structure, but a highly degenerate average over many structures.) On the other hand, it is a major unsolved problem to determine whether or not there is a uniquely ergodic tiling dynamical system (discrete or continuous) which has purely absolutely continuous spectrum [9], which we would call "chaotic", and which explains the title of this paper. The method of Theorem 1 cannot produce a strongly mixing **discrete** system since the factors, being substitution systems, cannot be strongly mixing [3,7]. It is unclear to us whether or not the method of Theorem 2 can produce a strongly mixing **continuous** system.

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