connected with the Casimir operators through

$$+r(r+1)+2+\sigma=2a,$$
$$+r(r+1)\sigma=a(a-1).$$

For all the representations considered in this paper we have $r = S = |k_0| = +(1-a)^{\frac{1}{2}}$. $S_{50} = \Gamma_0$ is given by (3.2):

$$S_{50}\,|a_v;\,k_0n_v;\,jj_3\rangle = \pm[(a-1+k_0^2)+(|k_0|+n)^2]^{\frac{1}{2}}\,|a_0;\,k_0n_0;\,jj_3\rangle. \quad \text{(A5)}$$

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[1] The characterization of a representation by a relation of the form (2.4) is very convenient. For Dirac representation of $SO(4, 2)$, for example, we have $S_{34} = -2S_{05}S_{12}$ or $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. Equation (2.4) is also a generalization of the relation $\{J_{ij}, J^i{}_k\} = ag_{jk}$ for $SU(2)$, which by virtue of the spectrum of J_3 gives only the 2-dimensional representation for $a = \frac{1}{2}$.

[2] T. D. Newton, Ann. Math. **51**, 730 (1950).

[3] T. Dixmier, Bull. Soc. Math. France **89**, 9 (1961).

[4] For example, A. Böhm, Nuovo Cimento **43**, 665 (1966), Appendix A. We adopt the convention that k_0 can be positive and negative and $c = k_0 + n$ positive only, so that $k_0 = j_1 - j_2$ and $|k_0| + n = j_1 + j_2 + 1$, where j_1 and j_2 are the characters of the $SO(3)^{(i)}$ in the Lie algebra decomposition $SO(4) = SO(3)^{(1)} \times SO(3)^{(2)}$.

[5] We can repeat the same considerations as above for the $SO(4, 1)_{51234}$ subgroup, replacing everywhere $S_{0\alpha}$ by S_{56}, and are led to the same conclusions for $S_{5\alpha}$ that we have for $S_{0\alpha}$.

[6] The irreducible representations of $SO(3, 1)_{0123}$ are characterized by two numbers (k_0', c) [M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon, New York, 1964)] so that $\frac{1}{2}S_{\mu\nu}S^{\mu\nu} = c^2 + k_0'^2 - 1$ and $\frac{1}{8}\varepsilon_{\mu\nu\lambda\delta}S^{\mu\nu}S^{\lambda\rho} = k_0'c$. Hence, from (2.21), spectrum $\Gamma_4{}^2 = -a + 1 - c^2 - k_0'^2$. Because k_0', is the lowest spin, $k_0' = k_0$. Hence, $k_0'^2 = 1 - a^2$ and therefore spectrum $\Gamma_4{}^2 = -c^2$. Because Γ_4 is a noncompact generator of a $SO(2, 1)$ subgroup, we know from general theorems that $-\infty <$ spectrum $\Gamma_4 < \infty$. Then spectrum $\Gamma_4{}^2 \geq 0$, and consequently $c^2 < 0$. Thus c is pure imaginary, and only the principal series representations (k_0', c) of $SO(3, 1)_{\mu\nu}$ appear in the reduction of our $SO(4, 2)$ representations with respect to $SO(3, 1)$. The spectrum of (ic) and therefore the reduction of these $SO(4, 2)$ representations with respect to $SO(3, 1)$ are continuous.

[7] J. B. Ehrman, Thesis, Princeton, 1954; in particular, Figs. 7-5, 7-7, and 7-15.

[8] J. Olszewski, Acta Phys. Polonica **30**, 105 (1966). See also I. Todorov, *Proceedings of the Coral Gables Conference on Fundamental Interactions* (Gordon & Breach, New York, 1969), Vol. I.

[9] A. Malkin and V. I. Man'ko, ZhETF Pis. Red. [JETP Lett.] **2**, 146 (1966); A. O. Barut and H. Kleinert, Phys. Rev. **156**, 1541 (1967); **157**, 1180 (1967); **160**, 1149 (1967); C. Fronsdal, *ibid.* **156**, 1665 (1967); Y. Nambu, Progr. Theoret. Phys. (Kyoto) Suppl. **37**, 368 (1966).

[10] A. O. Barut and H. Kleinert, Phys. Rev. **161**, 1464 (1967); H. Kleinert, *ibid.* **163**, 1807 (1967); A. O. Barut, D. Corrigan, and H. Kleinert, Phys. Rev. Letters **20**, 167 (1968); Phys. Rev. **167**, 1527 (1968); A. O. Barut, Phys. Letters **26B**, 308 (1968).

[11] A. O. Barut, in *Proceedings of the Coral Gables Conference on Fundamental Interactions* (Gordon & Breach, New York, 1970); Phys. Rev. (to be published).

[12] See, e.g., G. Mack and I. Todorov, ICTP, IC/68/86, Trieste, Italy; L. Gross, J. Math. Phys. **5**, 687 (1964).

[13] A. O. Barut, Nucl. Phys. **B4**, 455 (1968); A. O. Barut and K. C. Tripathy, Nucl. Phys. **B7**, 125 (1968).

[14] S. Ström, Arkiv Fysik **30**, 455 (1965).

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Approach to Equilibrium in a Simple Model

CHARLES RADIN
Department of Physics and Astronomy, The University of Rochester
Rochester, New York 14627

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The time evolution of a class of generalized quantum Ising models (with various long-range interactions, including Dyson's $1/r^\alpha$) has been studied from the C^*-algebraic point of view. We establish that: (1) All $\langle A\rangle_t$ are weakly almost periodic in time; (2) there exists a unique averaging procedure over time; (3) the time evolution in the thermodynamical limit can be locally implemented by effective Hamiltonians in the algebra of quasilocal observables; (4) there exists a specific connection between the spectral properties of the time evolution of the initial state and the approach to equilibrium; (5) there are examples in which the time evolution is not G-Abelian.

1. INTRODUCTION

A general class of Ising-type 2-body interactions on an infinite lattice of spins is considered, with the time behavior of the model being the object of study. It is shown that there is a canonical time average on the states of the system which gives a manageable prescription for determining the equilibria for time-developing states. Three subclasses of models are then singled out for a detailed study of the approach to equilibrium which they produce—finite-range interactions and infinite-range interactions decreasing polynomially and exponentially with respect to distance. With the results of a free-induction relaxation experiment in mind, a class of physically significant states is studied, yielding the corresponding equilibria and the detailed rate of approach to equilibrium for each type of interaction. Finally, the time behavior of each model is linked to specific spectral properties of the corresponding effective Hamiltonians.

The framework for the models is that of the C^*-algebra approach. The theories of invariant means on groups and asymptotic probability distributions are also employed.

2. THE GENERALIZED ISING MODEL

At each site i in a ν-dimensional lattice $\mathbf{Z}^\nu$, associate a 2-dimensional complex Euclidean space $\mathbf{C}_i^2$ (spin space). Let F be the set of all finite subsets of $\mathbf{Z}^\nu$. Then, for each finite volume $V \in F$, consider the direct product space $\otimes_{i\in V} \mathbf{C}_i^2$ of Murray and von Neumann,[1] and let $\mathfrak{A}(V)$ be defined as $B(\otimes_{i\in V} \mathbf{C}_i^2)$, the set of all bounded operators on $\otimes_{i\in V} \mathbf{C}_i^2$. $\mathfrak{A}(V)$ is the set of observables pertinent to the volume V. It is a concrete C^*-algebra with respect to the usual operator norm, denoted $\| \ \|$, and adjoint, denoted *. For any two volumes V, $V' \in F$, satisfying $V \subset V'$, there is a natural mathematical way to imbed $\mathfrak{A}(V)$ in $\mathfrak{A}(V')$—symbolically, for $A \in \mathfrak{A}(V)$ let $\tilde{A} = A \otimes_{i\in V'/V} I_i \in \mathfrak{A}(V')$, where I_i is the identity operator in $\mathfrak{A}(i)$ and $V'/V = \{j \in \mathbf{Z}^\nu \mid j \in V', j \notin V\}$. It is easy to check that this imbedding is norm preserving and, in fact, a $*$-isomorphism of the C^*-algebra $\mathfrak{A}(V)$ onto a subalgebra of $\mathfrak{A}(V')$. To obtain all local observables, we follow the prescription of Takeda,[2] which essentially involves the construction of a $*$-algebra $\mathfrak{A}^0$ defined as the union $\bigcup_{V\in F} \mathfrak{A}(V)$ with "equivalent" elements identified. This normed algebra is not complete, but upon completion it is a C^*-algebra, denoted $\mathfrak{A}$, consisting of the so-called quasilocal observables.[3] We denote by $\mathfrak{S}$ the set of all states on $\mathfrak{A}$.

To simplify notation throughout this paper, we identify $\mathfrak{A}(V)$ with its image in $\mathfrak{A}^0$ or $\mathfrak{A}$ and also with the matrix algebra $GL(2^{N(V)}, \mathbf{C})$, where $N(V)$ is the number of sites in V. For example, $\sigma_x^i \in \mathfrak{A}(i) \subset \mathfrak{A}^0 \subset \mathfrak{A}$ for the Pauli matrix σ_x.

We now turn our attention to the dynamics of the system.[4] With each $V \in F$, we associate an energy observable $H_V \in \mathfrak{A}(V)$ defined by

$$H_V = \tfrac{1}{2} \sum_{(j,k)\in V\times V} \epsilon_{jk} \sigma_z^j \sigma_z^k,$$

where $V \times V$ is the Cartesian product of V with itself. To make H_V self-adjoint, we require that ϵ_{jk} be real; for homogeneity and isotropy we require that ϵ_{jk} be a function only of the Euclidean distance $|j-k|$ between j and k, i.e., $\epsilon_{jk} = \epsilon(|j-k|)$. To avoid self-interaction, we assume that $\epsilon_{jj} = 0$ and, for stability, we require that the total energy at any site due to interaction with the entire lattice be finite, i.e., $\sum_{j\in\mathbf{Z}^\nu} |\epsilon(|j|)| < \infty$. We call this the generalized Ising model (GIM).

For $A \in \mathfrak{A}^0$, we can give the dynamics as follows. Consider $\alpha_t^V(A) = \exp(iH_V t) A \exp(-iH_V t)$. It is easy to see that the α_t^V are $*$-automorphisms of $\mathfrak{A}^0$ and form a group with the multiplication

$$(\alpha_{t_1}^V \cdot \alpha_{t_2}^V)(A) = \alpha_{t_1}^V(\alpha_{t_2}^V(A)) = \alpha_{t_1+t_2}^V(A),$$

i.e.,

$$\alpha_{t_1}^V \alpha_{t_2}^V = \alpha_{t_1+t_2}^V.$$

Clearly, we need to take the infinite volume limit to get the full dynamics. Therefore, for a local observable $A \in \mathfrak{A}(V_1)$, consider $\alpha_t^{V_2}(A)$ with $V_2 \supset V_1$. By Magnus' formula,[5] we have

$$\alpha_t^{V_2}(A) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} [H_{V_2}, [\cdots, [H_{V_2}, A]\cdots]]$$
$$= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \sum_{(j_1,k_1)\in V_2\times V_2} \cdots \sum_{(j_n,k_n)\in V_2\times V_2} \times [\Phi_{j_n k_n}, [\cdots, [\Phi_{j_1 k_1}, A]\cdots]],$$

where $\Phi_{jk} = \frac{1}{2}\epsilon_{jk}\sigma_z^j\sigma_z^k$ and [,] denotes the commutator. It is clear that, if j_1 and $k_1 \in V_2/V_1$, then $[\Phi_{j_1k_1}, A] = 0$. Therefore, we may restrict the relevant summation index to $(j_1, k_1) \in V_2 \times V_2/[(V_2/V_1) \times (V_2/V_1)]$. For any two subsets W_1 and W_2 of $\mathbf{Z}^\nu$, let $W_2 \div W_1$ denote $W_2 \times W_2/[(W_2/W_1) \times (W_2/W_1)]$, a subset of $\mathbf{Z}^\nu \times \mathbf{Z}^\nu$. Now consider $[\Phi_{j_1k_1}, A]$ in more detail. This operator can have at most σ_z's at the sites outside V_1. Therefore, if $(j_2, k_2) \in (V_2/V_1) \times (V_2/V_1)$, then $[\Phi_{j_2k_2}, [\Phi_{j_1k_1}, A]] = 0$. By induction, we see that we can restrict all n summation indices to $(j_i, k_i) \in V_2 \div V_1$. Now we bring these summation symbols back inside the brackets to get

$$\alpha_t^{V_2}(A) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \left[\sum_{(j_n,k_n)\in V_2\div V_1} \Phi_{j_nk_n}, \left[\cdots, \left[\sum_{(j_1,k_1)\in V_2\div V_1} \Phi_{j_1k_1}, A\right]\cdots\right]\right].$$

At this point, we take the infinite volume limit since, by the stability condition, the net $\sum_{(j,k)\in V_2\div V_1} \Phi_{jk}$ has a norm limit in $\mathfrak{A}$ as $V_2 \to \infty$, namely $\tilde{H}_{V_1} \equiv \sum_{(j,k)\in\mathbf{Z}^\nu\div V_1} \Phi_{jk}$. By Magnus' formula, we get

$$\text{norm-}\lim_{V_2\to\infty} \alpha_t^{V_2}(A) \equiv \alpha_t(A) = \exp(i\tilde{H}_{V_1}t)A \times \exp(-i\tilde{H}_{V_1}t).$$

To see that $\{\alpha_t^V(A) \mid V \in F\}$ is Cauchy for all $A \in \mathfrak{A}$, we use the inequality

$$\|\alpha_t^{V'}(A) - \alpha_t^V(A)\| \le \|\alpha_t^{V'}(A_0) - \alpha_t^V(A_0)\| + \|\alpha_t^{V'}(A - A_0)\| + \|\alpha_t^V(A_0 - A)\|.$$

Taking $A_0 \in \mathfrak{A}^0$ and using $\|\alpha_t^V\| = 1$ gives the result. At this point, it is easy to show that the α_t form a group of $*$-automorphisms of $\mathfrak{A}$. The fact that $\|\alpha_t(A) - \alpha_{t_0}(A)\| \underset{t \to t_0}{\to} 0$ for all $A \in \mathfrak{A}$ is easily checked (on $\mathfrak{A}^0$ first). We collect our results up to this point as

Proposition 1: If

$$H_V = \tfrac{1}{2} \sum_{(j,k) \in V \times V} \epsilon(|j-k|)\sigma_z^j \sigma_z^k, \qquad \epsilon(|j-k|)$$

is real, $\epsilon(|0|) = 0$, and $\sum_{j \in \mathbf{Z}^\nu} |\epsilon(|j|)| < \infty$, then, for all $A \in \mathfrak{A}$, the net $\alpha_t^V(A) = e^{iH_V t} A e^{-iH_V t}$ has a norm limit in $\mathfrak{A}$ as $V \to \infty$, denoted $\alpha_t(A)$. The set $\{\alpha_t \mid t \in \mathbf{R}\}$ forms a strongly continuous group of $*$-automorphisms of $\mathfrak{A}$ satisfying $\alpha_{t_1}\alpha_{t_2} = \alpha_{t_1+t_2}$. Furthermore, for any

$$A \in \mathfrak{A}(V), \quad \alpha_t(A) = e^{i\tilde{H}_V t} A e^{-i\tilde{H}_V t},$$

where

$$\tilde{H}_V \equiv \tfrac{1}{2} \sum_{(j,k) \in \mathbf{Z}^\nu \div V} \epsilon(|j-k|)\sigma_z^j \sigma_z^k.$$

As an application of this proposition, we obtain

$$\alpha_t(\sigma_x^j) = \sigma_x^j \cos(2P_j t) - \sigma_y \sin(2P_j t), \tag{1}$$

$$\alpha_t(\sigma_y^j) = \sigma_y^j \cos(2P_j t) + \sigma_x^j \sin(2P_j t), \tag{2}$$

where

$$P_j = \sum_{k \in \mathbf{Z}^\nu} \epsilon(|k-j|)\sigma_z^k.$$

3. EQUILIBRIUM

We now have a time development and wish to investigate the approach to equilibrium. A first step in this direction is to answer the following question: Given a nonequilibrium state ρ on $\mathfrak{A}$, what should be the corresponding equilibrium state $\tilde{\rho}$? A useful tool for investigating this problem is contained in a paper of Emch, Knops, and Verboven.[6] We first give some necessary background.

Let G be a topological group and define the normed linear space $CB(G)$ as the set of all bounded, continuous, complex-valued functions on G, with pointwise addition and scalar multiplication and sup norm. A mean on $CB(G)$ is by definition a linear form M on $CB(G)$ which satisfies

(i) $M(\bar{f}) = \overline{M(f)}$, where the overbar denotes complex conjugation for all f in $CB(G)$,

(ii) $\inf_{x \in G} |f(x)| \le M(f) \le \sup_{x \in G} |f(x)|$ for all real-valued f in $CB(G)$.

This is clearly a mathematical translation of the heuristic concept that M averages over the group.[7] M is called a left invariant mean if $M(L_y f) = M(f)$ for all $y \in G$ and all $f \in CB(G)$, where the translation $L_y f \in CB(G)$ is defined by $L_y f(x) = f(yx)$. If $\mathfrak{A}$ is a C^*-algebra with unit and if $\{\alpha_g \mid g \in G\}$ is a strongly continuous[8] group of $*$-automorphisms of $\mathfrak{A}$, then,[9] for any left invariant mean M on $CB(G)$ and state ρ on $\mathfrak{A}$, the form $M\rho$ defined on $\mathfrak{A}$ by $M\rho(A) = M[\rho(\alpha_x A)]$ is a state on $\mathfrak{A}$, invariant in the sense that $M\rho(\alpha_x A) = M\rho(A)$, for all $A \in \mathfrak{A}$, $x \in G$. Using $G = \mathbf{R}$ interpreted as time development, we see that $M\rho$ is a time average of the states $\alpha_t^* \rho$ defined by $\alpha_t^*\rho(A) = \rho(\alpha_t A)$. Therefore, each invariant mean[10] M could be used to project a given state onto possibly different equilibria. Since there are[11] many invariant means on $CB(\mathbf{R})$, the question of the uniqueness of this prescription arises. We now wish to investigate this question in the case of the GIM. To do so, we need some definitions.

The class $AP(\mathbf{R})$ of almost-periodic functions on the real line can be defined as the subset of $CB(\mathbf{R})$ of all f such that the set of translates $\{L_t f \mid t \in \mathbf{R}\}$ is precompact in the norm topology. The set of weakly almost-periodic functions $W(\mathbf{R})$ consists of the subset of $CB(\mathbf{R})$ of all f such that $\{L_t f \mid t \in \mathbf{R}\}$ is precompact in the weak topology. Since the weak topology is weaker than the norm topology, $AP(\mathbf{R}) \subset W(\mathbf{R})$. $W(\mathbf{R})$ plays an important role in the theory of invariant means since[12] all invariant means on $CB(\mathbf{R})$ coincide on the subspace $W(\mathbf{R})$; furthermore, they can be taken in the form

$$Mf = \lim_{T \to \infty} T^{-1} \int_0^T f(t)\, dt.$$

In this connection we now prove:

Proposition 2: In the GIM, $\rho[\alpha_t(A)]$ is a weakly almost-periodic function of $t \in \mathbf{R}$, for all $A \in \mathfrak{A}$ and $\rho \in \mathfrak{S}$.

Proof: First, let $A' = \sigma_{l_1}^{j_1} \cdots \sigma_{l_n}^{j_n}$, where $l_k = x, y$, or z, and all the j_k are distinct sites of the lattice. Then $\alpha_t(A') = \alpha_t(\sigma_{l_1}^{j_1}) \cdots \alpha_t(\sigma_{l_n}^{j_n})$. Using

$$\cos(B) = \tfrac{1}{2}[\exp(iB) + \exp(-iB)]$$

and

$$\sin(B) = [\exp(iB) - \exp(-iB)]/2i$$

in (1) and (2), where $B \in \mathfrak{A}$, we put $\alpha_t(A')$ in the form of a finite linear sum of terms such as

$$\exp(iQ_1 t)\sigma_{l_1}^{j_1} \cdots \exp(iQ_n t)\sigma_{l_n}^{j_n},$$

where, for $l_k = z$, $Q_k = 0$ and, otherwise, $Q_k = \pm 2P_{l_k}$. We move the exponentials to the right by noticing that

$$\exp(iQ_m t)\sigma_{l_k}^{j_k} = \sigma_{l_k}^{j_k} \exp(iR_m t),$$

where $R_m = \sigma_{l_k}^{j_k} Q_m \sigma_{l_k}^{j_k}$. Note that, independently of l_k, R_m only has σ_z's in it, i.e., $R_m = \sum_j a_j \sigma_z^j$ (a finite sum with $a_j \in \mathbf{R}$). Therefore, we also see that all the

R_m are self-adjoint and commute. By moving all the exponentials to the right in this way, we obtain $\alpha_t(A')$ as a finite linear sum of terms of the form $\sigma_{l_1}^{j_1} \cdots \sigma_{l_n}^{j_n} \exp(iSt)$ with S self-adjoint. If $\rho \in \mathfrak{S}$, by considering the GNS representation Π_ρ associated with ρ, with cyclic vector Φ_ρ in $\mathcal{H}_\rho$, we see that $\rho[\alpha_t(A')]$ is a finite linear sum of functions of t of the form

$$(\Phi_\rho, \Pi_\rho[\sigma_{l_1}^{j_1}] \cdots \Pi_\rho[\sigma_{l_n}^{j_n}] \exp(i\Pi_\rho[S]t)\Phi_\rho)$$

which, upon taking adjoints, becomes

$$(\Psi_\rho, \exp(i\Pi_\rho[S]t)\Phi_\rho).$$

But

$$(\Psi_\rho, \exp(i\Pi_\rho[S]t)\Phi_\rho) \in W(\mathbf{R}),$$

since[13] $\{\exp(i\Pi_\rho[S]t) \mid t \in \mathbf{R}\}$ is precompact in the weak operator topology. Therefore, $\rho[\alpha_t(A')] \in W(\mathbf{R})$, since $W(\mathbf{R})$ is a linear space. For an A'' equal to a finite linear sum of A's of the above form, the same result follows by linearity again. For arbitrary $A \in \mathfrak{A}$, take $A_n \to A$, as $n \to \infty$ (norm topology), with A_n of the latter form. Then we have

$$\|\alpha_t(A_n) - \alpha_t(A)\| = \|\alpha_t(A_n - A)\| = \|A_n - A\|.$$

Hence, $\rho[\alpha_t(A_n - A)] \to 0$, as $n \to \infty$, uniformly in $t \in \mathbf{R}$. Therefore, $\rho[\alpha_t(A)]$ is the limit of a sequence $\rho[\alpha_t(A_n)]$ of functions in $W(\mathbf{R})$, converging in the sup norm. Hence, $\rho[\alpha_t(A)] \in W(\mathbf{R})$, since[14] $W(\mathbf{R})$ is a closed subspace of $CB(\mathbf{R})$. QED

Note that a shorter proof of Proposition 2, which does not rely on the local implementation of Proposition 1, can be obtained by observing that

$$\begin{aligned}\exp(iH_Vt)A'\exp(-iH_Vt) &= A'\exp(iA'H_VA't)\\ &\quad\times\exp(-iH_Vt)\\ &= A'\exp(iA'H_VA't\\ &\quad - iH_Vt).\end{aligned}$$

However, parts of the given proof are needed below.

We have proven that a canonical time average[15] $M\rho$ exists for every initial state ρ. We use "canonical" to emphasize the uniqueness of M. Given this association between states and equilibrium states, we consider the following question: Can one ensure that $M\rho'$ will be "close" to $M\rho$ by taking ρ' sufficiently "close" to ρ? We answer this for the three simplest topologies.

Lemma 1: In the GIM, with $\mathfrak{S}$ in its norm topology, the mapping $M:\mathfrak{S} \to \mathfrak{S}$ is continuous.

Proof: Let ϕ, $\phi_n \in \mathfrak{S}$, with $\phi_n \to \phi$, as $n \to \infty$, in norm. For $x \in \mathfrak{A}_1$, the closed unit ball of $\mathfrak{A}$,

$$\phi_n[\alpha_t(x)] \to \phi[\alpha_t(x)],$$

as $n \to \infty$, uniformly in $t \in \mathbf{R}$ and uniformly in $x \in \mathfrak{A}_1$. Therefore, given $\epsilon > 0$, there exists $N > 0$ such that, for all $n \geq N$,

$$|\phi_n[\alpha_t(x)] - \phi[\alpha_t(x)]| < \epsilon, \quad \text{for all } t \in \mathbf{R},\ x \in \mathfrak{A}_1.$$

Therefore,

$$\left|\frac{1}{T}\int_0^T \phi_n[\alpha_t(x)]\,dt - \frac{1}{T}\int_0^T \phi[\alpha_t(x)]\,dt\right| \leq \frac{1}{T}\int_0^T \epsilon\,dt = \epsilon,$$

independently of $T > 0$. Hence, $M\phi_n \to M\phi$, as $n \to \infty$, in norm, i.e.,

$$\lim_{n\to\infty}\sup_{x\in\mathfrak{A}_1}\left|\lim_{T\to\infty}\frac{1}{T}\int_0^T(\phi_n[\alpha_t(x)] - \phi[\alpha_t(x)])\,dt\right| = 0.$$

QED

Corollary: In the GIM, with $\mathfrak{S}$ in its weak topology, the mapping $M:\mathfrak{S} \to \mathfrak{S}$ is continuous.

The proof is immediate from Dunford and Schwartz.[16]

Proposition 3: In the GIM with dimension $\nu = 1$, let $\epsilon(|j|) = 1/\xi^{|j|}$ for $j \neq 0$, $\xi > 2$. Then, with $\mathfrak{S}$ in its w^*-topology, the mapping $M:\mathfrak{S} \to \mathfrak{S}$ is not continuous.

Proof: A proof by contradiction is immediate from the following two facts: On the one hand, we exhibit a state ϕ such that $M\phi(\sigma_x^0) \neq 0$; on the other hand, we exhibit a subset U of $\mathfrak{S}$ which is w^*-dense and for which $M\rho(\sigma_x^0) = 0$ for all ρ in U. To this end, let ρ be the product state $\otimes_{j\in\mathbf{Z}} \hat{f}_j$, where $\hat{f}_j$ is the state on $\mathfrak{A}(j)$ defined by any normalized vector f_j which satisfies

$$\begin{aligned}\sigma_z^j f_j &= f_j, &&\text{if } j > 0,\\ \sigma_z^j f_j &= -f_j, &&\text{if } j < 0,\\ \sigma_x^0 f_0 &= f_0.\end{aligned}$$

Now let $\phi = M\rho$. Then, ϕ is time invariant and

$$\begin{aligned}\phi(\sigma_x^0) &= M[\rho(\alpha_t[\sigma_x^0])]\\ &= M[\rho(\sigma_x^0\cos[2P_0t] - \sigma_y^0\sin[2P_0t])].\end{aligned}$$

If Π_ρ is the GNS representation of $\mathfrak{A}$ corresponding to ρ, with cyclic vector Φ_ρ, it is easy to see that $\Pi_\rho(P_0)\Phi_\rho = 0$ from cancellations. Hence, $\rho(P_0^n) = 0$ for all $n \in \mathbf{N}$, $n \neq 0$. Therefore, $\rho[\alpha_t(\sigma_x^0)] = \rho(\sigma_x^0) = 1$ for all $t \in \mathbf{R}$. Hence, $M\phi(\sigma_x^0) = \phi(\sigma_x^0) = 1$. This concludes the first part of the proof. Now, from Dixmier,[17] we know that the set U of vector states of any nonnull representation of the simple, antiliminal[18] C^*-algebra $\mathfrak{A}$ is w^*-dense in $\mathfrak{S}$. Consider the GNS representation generated by $\rho = \otimes_{j\in\mathbf{Z}} \hat{f}_j$, where $\sigma_z^j f_j = f_j$ for all $j \in \mathbf{Z}$. Choose the orthonormal basis for $\mathcal{H}_\rho$ consisting of $\{\Psi_I \mid I \in F\}$, where

$$\Psi_I \equiv \sum_{i\in I} \Pi_\rho(\sigma_x^i)\Phi_\rho \quad \text{for} \quad I \in F, \quad I \neq \varnothing,$$

and

$$\Psi_\varnothing \equiv \Phi_\rho .$$

Now $\alpha_t(\sigma_x^0) = \sigma_x^0 \cos(2P_0 t) - \sigma_y^0 \sin(2P_0 t)$. Note that $\Pi_\rho(P_0)\Psi_I = p_I \Psi_I$, where $p_I \equiv \sum_{j\in\mathbf{Z}} \epsilon(|j|) g_j$ and

$$g_j = +1, \quad \text{if} \quad j \in \mathbf{Z}/I,$$
$$= -1, \quad \text{if} \quad j \in I.$$

Therefore,

$$\Pi_\rho(P_0^m)\Psi_I = (p_I)^m \Psi_I$$

and

$$(\Psi_I, \Pi_\rho(P_0^m)\Psi_J) = \delta_{I,J} p_I^m .$$

If Ω is a unit vector in $\mathcal{H}_\rho$, let $\Omega = \sum_{I\in F} \omega_I \Psi_I$. Hence,

$$(\Omega, \Pi_\rho(P_0^m)\Omega) = \sum_{I,J\in F} \bar{\omega}_I \omega_J (\Psi_I, \Pi_\rho(P_0^m)\Psi_J) = \sum_{I\in F} \bar{\omega}_I \omega_I p_I^m .$$

Similarly,

$$(\Omega, \Pi_\rho[\sigma_x^0 P_0^m]) = \sum_{I\in F} \bar{\omega}_{I_0} \omega_I p_I^m$$

where $I_0 = I/\{0\}$ if $0 \in I$, $I_0 = I \cup \{0\}$ if $0 \notin I$. Therefore,

$$(\Omega, \Pi_\rho(\sigma_x^0) \cos[2\Pi_\rho(P_0)t]\Omega) = \left(\Omega, \sum_{n=0}^{\infty} \frac{(it)^{2n}}{(2n)!} \Pi_\rho(\sigma_x^0 P_0^{2n})\Omega\right) = \sum_{I\in F} \bar{\omega}_{I_0} \omega_I \cos(2p_I t).$$

By taking finite sums in $I \in F$, we can approximate $h(t) \equiv (\Omega, \Pi_\rho(\sigma_x^0) \cos[2\Pi_\rho(P_0)t]\Omega)$ uniformly in $t \in \mathbf{R}$. On the finite sums $h_N(t)$, since no p_I can vanish for the given interactions, we have $Mh_N = 0$. Therefore, $Mh = 0$, since M is continuous on $CB(\mathbf{R})$. Similarly, $M[(\Omega, \Pi_\rho(\sigma_y^0) \sin(2\Pi_\rho[P_0]t)\Omega)] = 0$, so that

$$[M\hat{\Omega}](\sigma_x^0) = 0$$

for all vector states $\hat{\Omega}$ corresponding to the representation Π_ρ. QED

Note that the above proof can also be used to show that the GIM is not always G-Abelian in time, i.e., need not satisfy the condition

$$M[\phi([\alpha_t(A), B])] = 0 \quad \text{for all } A, B \in \mathfrak{A} \text{ and all } \phi \in \mathfrak{S}_I,$$

where $\mathfrak{S}_I$ is the set of time invariant states. To see this, just use the ϕ of the above proof and $A = \sigma_z^0$, $B = \sigma_y^0$. This example reinforces the doubts one might have of the validity of the assumption that general systems are G-Abelian in time and, hence, justifies the attempt to avoid the assumption. Compare in this respect Knops[19] and Emch, Knops, and Verboven.[20] See also Araki.[21]

Proposition 3 is rather disconcerting in that one has good reason for taking the w^*-topology on $\mathfrak{S}$ as the most physical one. The proposition might, however, be an indication of the fact that $\mathfrak{S}$ itself is bigger than actually needed for physical purposes.[22] It is, furthermore, conceivable that M is w^*-continuous on a w^*-dense subset $\mathfrak{S}_0$ of $\mathfrak{S}$, where $\mathfrak{S}_0$ itself contains all physically accessible states.

4. TEMPERATURE STATES

It is desirable, for the consistency of the approach used in this paper, to establish the existence of infinite volume limits of the usual canonical equilibrium ensembles since, in the present theory, these limits should play the role of states describable by a temperature. Specifically, the question is whether one can take a limit of the states[23] ρ_β^V defined on $\mathfrak{A}(V)$ by

$$\rho_\beta^V(A) = \mathrm{Tr}_V(A \exp[-\beta H_V])/\mathrm{Tr}_V(\exp[-\beta H_V]),$$

where Tr_V is the usual normalized trace state on $\mathfrak{A}(V)$. Araki has shown that, for a 1-dimensional lattice and any finite-range interaction, such a limit does exist.[24] By restricting ourselves to ferromagnetic Ising-type interactions, we obtain the same conclusion for infinite-range interactions in ν dimensions.[25]

Proposition 4: In the GIM, assume that $\epsilon(|j|) \leq 0$ for all $j \in \mathbf{Z}^\nu$. Then, extending the canonical ensemble ρ_β^V to the state $\tilde{\rho}_\beta^V \equiv \rho_\beta^V \otimes_{j\in\mathbf{Z}^\nu/V} \mathrm{Tr}_j$ on $\mathfrak{A}$, we see that there exists a state ρ_β^∞ on $\mathfrak{A}$ defined as $w^*\text{-}\lim_{V\to\infty} \tilde{\rho}_\beta^V$.

Proof: The proof consists of reducing the problem to the finite-volume subalgebras where generalized Griffiths inequalities can be used. Consider, then, any three nonempty elements V_1, V_2, and V_3 of F such that $V_1 \subset V_2 \subset V_3$. Introduce the following two interactions on $\mathfrak{A}(V_3)$:

$$H_3 = \tfrac{1}{2} \sum_{(j,k)\in V_3\times V_3} \epsilon(|j-k|)\sigma_z^j \sigma_z^k$$

and

$$H_2 = \tfrac{1}{2} \sum_{(j,k)\in V_2\times V_2} \epsilon(|j-k|)\sigma_z^j \sigma_z^k .$$

Note that H_3 can be obtained from H_2 by adding ferromagnetic bonds. If, for each $\rho \in \mathfrak{S}$, we define the state $\rho \mid \mathfrak{A}(V)$ on $\mathfrak{A}(V)$ by restriction, then we have

$$\tilde{\rho}_\beta^{V_3}(\cdot) \mid \mathfrak{A}(V_1) = \mathrm{Tr}_{V_3}(\cdot \exp[-\beta H_3])/\mathrm{Tr}_{V_3}(\exp[-\beta H_3]) \mid \mathfrak{A}(V_1)$$

and

$$\tilde{\rho}_\beta^{V_2}(\cdot) \mid \mathfrak{A}(V_1) = \mathrm{Tr}_{V_2}(\cdot \exp[-\beta H_2])/\mathrm{Tr}_{V_2}(\exp[-\beta H_2]) \mid \mathfrak{A}(V_1).$$

We now show that, in fact,

$$\tilde{\rho}_\beta^{V_2}(\cdot) \mid \mathfrak{A}(V_1) = \mathrm{Tr}_{V_3}(\cdot \exp[-\beta H_2])/\mathrm{Tr}_{V_3}(\exp[-\beta H_2]) \mid \mathfrak{A}(V_1). \tag{3}$$

To see this, introduce the following orthonormal basis in $\bigotimes_{i\in V_3} \mathbf{C}_i^2$:

$$\begin{aligned} e_1 &= f_1^1 \otimes f_2^1 \otimes \cdots \otimes f_{N(V_3)}^1, \\ e_2 &= f_1^2 \otimes f_2^1 \otimes \cdots \otimes f_{N(V_3)}^1, \\ e_3 &= f_1^1 \otimes f_2^2 \otimes \cdots \otimes f_{N(V_3)}^1, \\ e_4 &= f_1^2 \otimes f_2^2 \otimes \cdots \otimes f_{N(V_3)}^1, \\ &\ \ \vdots \\ e_{2^{N(V_3)}} &= f_1^2 \otimes f_2^2 \otimes \cdots \otimes f_{N(V_3)}^2, \end{aligned}$$

where f_i^j is a fixed normalized vector in $\mathbf{C}_i^2$ satisfying $\sigma_z^i f_i^1 = f_i^1$ and $\sigma_z^i f_i^2 = -f_i^2$. If $A \in \mathfrak{A}(V_1)$, then $\exp(-\beta H_2)A \in \mathfrak{A}(V_2)$. Calculating in the above basis gives, for any $B \in \mathfrak{A}(V_2)$, we obtain

$$\mathrm{Tr}_{V_3}(B) = 2^{N(V_3)-N(V_2)}\,\mathrm{Tr}_{V_2}(B).$$

Therefore, (3) follows. Now we need some further notation before we can continue with the proof. For each triple $A = (A_1, A_2, A_3)$ where the $A_i \in F$ are pairwise disjoint, define σ^A as $(\prod_{i\in A_1}\sigma_x^i)(\prod_{j\in A_2}\sigma_y^j) \times (\prod_{k\in A_3}\sigma_z^k)$, where $\prod_{\ell\in\varnothing} B_\ell$ is defined to be the identity I. Note that $\mathfrak{A}^0$ is the linear manifold generated by the set of all σ^A. Now, if $A_1 \cup A_2 \neq \varnothing$, then

$$\tilde{\rho}_\beta^V(\sigma^A) = 0 \tag{4}$$

for all $V \in F$ such that $V \supset A_1 \cup A_2 \cup A_3$ since in the above basis the diagonal elements of $\exp(-\beta H_V)\sigma^A$ are all zero. Kelly and Sherman[26] have shown that, by increasing the number of bonds, we have

$$\tilde{\rho}_\beta^{V_3}(\sigma^A) \geq \tilde{\rho}_\beta^{V_2}(\sigma^A)$$

if

$$A = (\varnothing, \varnothing, A_3) \quad \text{and} \quad A_3 \subset V_2 \subset V_3. \tag{5}$$

Combining (4) and (5) with another theorem of Kelly and Sherman, which says that $\tilde{\rho}_\beta^{V'}(\sigma^A) \geq 0$ if $\sigma^A \in \mathfrak{A}(V) \subset \mathfrak{A}(V')$ and $A = (\varnothing, \varnothing, A_3)$, we have, for $V' \supset V$ and $\sigma^A \in \mathfrak{A}(V)$,

$$0 \leq \tilde{\rho}_\beta^{V'}(\sigma^A) \leq 1 \quad \text{and} \quad \tilde{\rho}_\beta^{V'}(\sigma^A) \geq \tilde{\rho}_\beta^{V}(\sigma^A),$$

so that $\tilde{\rho}_\beta^V(\sigma^A)$ is an increasing function of V, bounded above by 1. Therefore, $\lim \tilde{\rho}_\beta^V(\sigma^A)$, as $V \to \infty$, exists if V increases by inclusion. Define $\rho_\beta^\infty(\sigma^A)$ as the limit. Considering ρ_β^∞ as a functional on the $*$-subalgebra $\mathfrak{A}^0$ of $\mathfrak{A}$, ρ_β^∞ is clearly linear and bounded, with norm 1. When ρ_β^∞ is extended to $\mathfrak{A}$, it still has norm 1 and satisfies $\rho_\beta^\infty(I) = 1$. Therefore, it is a state on $\mathfrak{A}$.

QED

5. APPROACH TO EQUILIBRIUM

Now that we have shown the existence of a canonical association between arbitrary initial states and equilibrium states, and also that at least some of these equilibrium states are reasonable, we would like to investigate the association in more detail. One reason for this is to examine the question of recurrences. We are motivated in this approach by a paper of Emch,[27] where the following experiment is considered.

A CaF_2 crystal is placed in a magnetic field (thus determining the z direction), and allowed to reach thermal equilibrium. Then, an rf pulse is applied which turns the net nuclear magnetization to the x direction. The magnetization in the x direction is then measured as a function of time, and the result is an oscillatory function which damps to the equilibrium value of zero.[28]

Emch assumes an interaction of the form

$$H_V = \Big(\tfrac{1}{2}\sum_{(j,k)\in V\times V} \epsilon(|j-k|)\sigma_z^j\sigma_z^k\Big) - B\sum_{j\in V}\sigma_z^j$$

on a finite 1-dimensional volume V. As the state representing the system after the application of the rf pulse, he takes the product state $\rho = \bigotimes_{j\in V}\phi_j$, where

$$\phi_j(\cdot) = \mathrm{Tr}_j(\cdot \exp[-\gamma\sigma_x^j])/\mathrm{Tr}_j(\exp[-\gamma\sigma_x^j]).$$

This choice is justified by an entropy argument. With the interaction H_V and initial state ρ, he then calculates the time development of the magnetization in the x direction, $S_x = [1/N(V)]\sum_{j\in V}\sigma_x^j$, and obtains without approximation

$$\rho(\alpha_t^V[S_x]) = \rho(S_x)\Big(\prod_{j\in V}\cos^2 2\epsilon(|j|)t\Big)\cos 2Bt. \tag{6}$$

By taking an infinite volume limit at this point, Emch shows that interactions with a cutoff give recurrences (with calculable frequency) and that the infinite range interaction of the form $\epsilon(|j|) = 1/2^{|j|}$ gives, with Vieta's identity,[29]

$$\prod_{n=1}^{\infty}\cos\Big(\frac{t}{2^n}\Big) = \frac{(\sin t)}{t},$$

the nonrecurrent damping exhibited by experiment.

In this section, we generalize this work in the following respects. We consider arbitrary observables and show what range of behavior is possible with different choices of the function ϵ. Furthermore, the class of initial states considered is extended, and the approach to equilibrium is exhibited in the stronger form of an initial state decaying into an equilibrium state, rather than just considering individual expectation values. To simplify calculations, we assume throughout that there is no external field. Inspection of (6) shows that the damping we are looking for comes solely from the spin–spin interaction of the lattice, not the external field.

The general result for finite range models, hereafter denoted F_L, is then:

Lemma 2: In the GIM, if $\epsilon(|j|) = 0$ for all $j \in \mathbf{Z}^\nu$ such that $|j| > L$, $0 < L < \infty$, then $\rho[\alpha_t(A)]$ is almost periodic for all $A \in \mathfrak{A}$, $\rho \in \mathfrak{S}$.

Proof: From the proof of Proposition 2 with the above hypothesis added, we see that the observable S is only a finite linear sum of σ_z's. This implies that the spectrum of S, and hence of $\Pi_\rho(S)$, consists of a finite number of isolated points. Using the spectral theorem in $\mathcal{H}_\rho$, we get $[\alpha_t(A')]$ in the form

$$\sum_{j=1}^{N} a_j e^{ib_j t}, \quad b_j \in \mathbf{R},$$

which is almost periodic. Since $AP(\mathbf{R})$ is also a closed linear subspace of $CB(\mathbf{R})$, we get the result for all $A \in \mathfrak{A}$ as in Proposition 3. QED

It will become apparent later that the full range of behavior of the GIM due to different choices of the function ϵ is already predictable from (6). Because of its importance, therefore, we derive a convenient generalization of (6).

Let ϕ be any state which satisfies $\phi(\sigma^A) = 0$ for all A such that $A_3 \neq \varnothing$. Consider $\phi[\alpha_t(\sigma_x^0)]$ in the GIM. As shown in Proposition 1,

$$\alpha_t(\sigma_x^0) = \operatorname*{norm-lim}_{V\to\infty} \alpha_t^V(\sigma_x^0), \quad \text{where} \quad V \in F,$$

$$\alpha_t^V(\sigma_x^0) = \sigma_0^x \cos\left(2t\sum_{j\in V} R_j\right) - \sigma_y^0 \sin\left(2t\sum_{j\in V} R_j\right),$$

and $R_j = \epsilon(|j|)\sigma_z^j$. Using the conditions on ϕ, we have

$$\phi[\alpha_t^V(\sigma_x^0)] = \phi\left[\sigma_x^0 \cos\left(2t\sum_{j\in V} R_j\right)\right]. \tag{7}$$

We show by induction on the number of sites in V that

$$\phi[\alpha_t^V(\sigma_x^0)] = \phi(\sigma_x^0)\prod_{i\in V}\cos[2\epsilon(|j|)t]. \tag{8}$$

For $V = \{j\}$, $\cos(2tR_j) = \cos[2t\epsilon(|j|)]$, since $(\sigma_z^j)^{2n} = 1$ for all $n \in \mathbf{N}$ and (8) follows from (7). Now assume (8) for V having N sites with site $l \notin V$. Then

$$\phi\left\{\sigma_x^0 \cos\left[2t\left(\sum_{j\in V} R_j + R_l\right)\right]\right\} = \phi\left[\sigma_x^0 \cos\left(2t\sum_{j\in V} R_j\right)\cos(tR_l) - \sigma_x^0 \sin\left(2t\sum_{j\in V} R_j\right)\sin(2tR_l)\right].$$

The second term on the rhs vanishes since, in the series expansion of the sines, every term has at least one "unmatched" σ_z in it which is annihilated by ϕ. Again, $\cos(2tR_l) = \cos[2t\epsilon(|l|)]$, so that

$$\phi[\alpha_t^{V\cup l}(\sigma_x^0)] = \phi(\sigma_x^0)\cdot\prod_{j\in V\cup l}\cos[2\epsilon(|j|)t].$$

By induction, we have (8) for all $V \in F$, $V \neq \varnothing$. To take the volume limit, we first define $\prod_{j\in\mathbf{Z}^\nu} a_j$, where $a_j \in \mathbf{C}$, as the limit, if it exists, of the net $\prod_{j\in V} a_j$. We make no exceptions for zero factors or convergence to zero. Since $\sum_{j\in\mathbf{Z}^\nu} |\epsilon(|j|)| < \infty$, we must have $\epsilon(|j|) \xrightarrow[j\to\infty]{} 0$, so that it is clear that the limit exists[30] for the net $\prod_{j\in V}\cos[2t\epsilon(|j|)]$. Hence,

$$\phi[\alpha_t(\sigma_x^0)] = \phi(\sigma_x^0)\prod_{j\in\mathbf{Z}^\nu}\cos[2\epsilon(|j|)t]. \tag{6'}$$

A natural means of investigating the influence of a particular choice of ϵ is thus determining the resulting behavior of $\prod_{n=1}^\infty \cos[\epsilon(n)t]$. As mentioned above, Vieta's formula shows that the choice $\epsilon(|j|) = 1/2^{|j|}$ produces nonrecurrent behavior. More generally, one might inquire into the time behavior resulting from $\epsilon(|j|) = 1/\xi^{|j|}$, $\xi > 1$. The model with $\epsilon(|j|) = (\xi^{|j|})^{-1}$ for $j \neq 0$, where $\xi > 1$, for stability, is called the exponential model E_ξ. We have[31] that

$$\prod_{n=1}^\infty \cos\left(\frac{t}{\xi^n}\right) \xrightarrow[t\to\infty]{} 0, \quad \text{for} \quad \xi > 1,$$

$$\text{if and only if } \xi \notin S/\{2\}, \tag{9}$$

where S is the countable set of all algebraic integers over the rationals with conjugates having moduli strictly less than one. From this, we see that the qualitative behavior of the model is discontinuous in ξ. For this reason, and because of certain results concerning phase transitions by Dyson,[32] we also consider the following form for ϵ: $\epsilon(|j|) = 1/j^{|\alpha|}$ for $j \neq 0$, where α is assumed greater than the dimension ν for stability. We call this the Dyson model D_α. The nonrecurrent time behavior of the Dyson models is shown by the following lemma.

Lemma 3:

$$\prod_{j=1}^{\infty} \cos^2\left(\frac{t}{j^\alpha}\right) \xrightarrow[t\to\infty]{} 0, \quad \text{for all} \quad \alpha > 1.$$

Proof: Define

$$f(t) = \prod_{j=1}^{\infty} \cos^2\left(\frac{t}{j^\alpha}\right) = \prod_{j=1}^{\infty}\left[1 - \sin^2\left(\frac{t}{j^\alpha}\right)\right].$$

Assume that $f(t)$ does not have limit zero as $t \to \infty$. Then there exists a $\delta > 0$ and a sequence $t_n > 0$ such that $t_n \to \infty$ and $f(t_n) > \delta$ for all $n \in \mathbf{N}$. Now $e^x \geq 1 + x$ for all $x \in \mathbf{R}$. Therefore, if $1 + x_j \geq 0$,

$$\prod_{j=1}^{N} e^{x_j} \geq \prod_{j=1}^{N} (1 + x_j).$$

Therefore,

$$\prod_{j=1}^{\infty} \exp\left[-\sin^2\left(\frac{t_n}{j^\alpha}\right)\right] \geq \delta \quad \text{for all } n.$$

Taking logarithms, we obtain

$$-\sum_{j=1}^{\infty} \sin^2\left(\frac{t_n}{j^\alpha}\right) \geq \ln \delta.$$

Hence,

$$\sum_{j=1}^{\infty} \sin^2\left(\frac{t_n}{j^\alpha}\right) \leq -\ln \delta, \quad \text{for all} \quad n \in \mathbf{N}. \tag{10}$$

Let N_n be the number of solutions m in N of the expression $\sin^2(t_n/m^\alpha) \geq \frac{1}{4}$. Clearly, N_n is greater than or equal to the number of solutions m in N of

$$\tfrac{1}{6}\pi \leq t_n/m^\alpha \leq \tfrac{5}{6}\pi$$

or

$$m(\tfrac{1}{6}\pi)^{1/\alpha} \leq (t_n)^{1/\alpha} \leq m(\tfrac{5}{6}\pi)^{1/\alpha}. \tag{11}$$

Therefore, $N_n \to \infty$ as $t_n \to \infty$. This contradicts (10). Hence, $f(t) \to 0$ as $t \to \infty$. QED

We now combine the above facts to prove the following proposition.

Proposition 5: With dimension $\nu = 1$, let the interaction be that of any exponential model E_ξ, where ξ is transcendental, or any Dyson model. Let ϕ be any state which satisfies $\phi(\sigma^A) = 0$ for all A such that $A_3 \neq \varnothing$. Then $M\phi = \bigotimes_{j\in\mathbf{Z}} \mathrm{Tr}_j$.

Proof: We show that $M\phi$ and $\bigotimes_{j\in\mathbf{Z}} \mathrm{Tr}_j$ coincide on the set of all σ^A, which by linearity and continuity will give the full result. Note that $\bigotimes_{j\in\mathbf{Z}} \mathrm{Tr}_j(\sigma^A) = 0$ for all $A \neq (\varnothing, \varnothing, \varnothing)$. For $A = (\varnothing, \varnothing, A_3)$, $\alpha_t(\sigma^A) = \sigma^A$ so that $\phi[\alpha_t(\sigma^A)] = \phi(\sigma^A)$, and the coincidence is obvious. Hence, for the rest of the proof, we assume that $A_1 \cup A_2 \neq \varnothing$. As in the proof of Proposition 2, $\alpha_t(\sigma^A)$ can be put in the form of a finite linear sum of terms, such as

$$\sigma_z^{j_1} \cdots \sigma_z^{j_k} \sigma_{l_{k+1}}^{j_{k+1}} \cdots \sigma_{l_{k+n}}^{j_{k+n}} \exp(iSt),$$

where $S = \sum_{j\in\mathbf{Z}} a_j \sigma_z^j$, and for $j \in \mathbf{Z}$ such that $|j| > W \equiv \max_{m=1,\cdots,n}\{|j_{k+m}|\}$ we have

$$\tfrac{1}{2}a_j = \pm\epsilon(|j - j_{k+1}|) \pm \cdots \pm \epsilon(|j - j_{k+n}|).$$

To show that $M\phi(\sigma^A) = 0$, we first notice that $M(\sigma^A)$ is a finite linear sum of terms of the form

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T (\Pi_\phi[\sigma_{l_{k+n}}^{j_{k+n}} \cdots \sigma_z^{j_1}]\Phi_\phi, \exp[i\Pi_\phi(S)t]\Phi_\phi)\, dt.$$

By von Neumann's ergodic theorem,[33] the above expression equals $(\Psi_\phi, P\Phi_\phi)$, where

$$\Psi_\phi \equiv \Pi_\phi[\sigma_{l_{k+n}}^{j_{k+n}} \cdots \sigma_z^{j_1}]\Phi_\phi$$

and P is the projection defined by the strong operator limit,

$$\lim_{\epsilon\to 0-}[E(0) - E(\epsilon)],$$

where $\{E(\lambda) \mid \lambda \in \mathbf{R}\}$ is the resolution of the identity corresponding to $\Pi_\phi(S)$. We will show that $P\Phi_\phi = 0$, and this will complete the proof:

$$\|P\Phi_\phi\| = (\Phi_\phi, P\Phi_\phi) = \lim_{T\to\infty} T^{-1}\int_0^T (\Phi_\phi, \exp[i\Pi_\phi(S)t]\Phi_\phi)\, dt.$$

By arguing as in the proof of (6′), we get

$$(\Phi_\phi, \exp[i\Pi_\phi(S)t]\Phi_\phi) = \prod_{j\in\mathbf{Z}} \cos(a_j t).$$

All we need to do now is show that

$$\prod_{j\in\mathbf{Z}} \cos(a_j t) \xrightarrow[t\to\infty]{} 0. \tag{12}$$

For the exponential models, we have for all large enough $|j|$ that

$$\tfrac{1}{2}a_j = \pm\frac{1}{\xi^{|j-j_1|}} \pm \cdots \pm \frac{1}{\xi^{|j-j_n|}}.$$

Therefore, if ξ is transcendental, it is clear that a_j does not vanish. Then,since a_j can be factored,

$$\tfrac{1}{2}a_j = \frac{1}{\xi^j}[\pm\xi^{j_1} \pm \cdots \pm \xi^{j_n}], \qquad \text{for} \quad j \gg 0,$$
$$= \frac{1}{\xi^{-j}}[\pm\xi^{-j_1} \pm \cdots \pm \xi^{-j_n}], \quad \text{for} \quad j \ll 0,$$

a simple change of variable in (9) gives (12). For the Dyson models, we use a different argument. First, we need to show that a_j does not vanish for all sufficiently large $|j|$. There are two cases to be treated separately: (1) a_j does not have an equal number of $+2$ and -2 coefficients for the $1/|j - j_1|^\alpha$; (2) it does have an equal number. In case 1, it is clear that the sign which appears more often gives a_j that same sign for all large enough $|j|$. Case 2 follows from the fact that the functions of a complex variable defined by

$$a_1(z) = \pm \frac{1}{(z - j_1)^\alpha} \pm \cdots \pm \frac{1}{(z - j_n)^\alpha}$$

and

$$a_2(z) = \pm \frac{1}{(j_1 - z)^\alpha} \pm \cdots \pm \frac{1}{(j_n - z)^\alpha}$$

are both analytic at infinity, so that there is a compact set K which contains the zeros of both functions. An argument similar to Lemma 3 then yields (12).

QED

In addition to the result stated in the proposition, we point out that the above proof not only determined the equilibrium value of all local observables, but also showed that this equilibrium value is actually approached for large t. In fact, we have proven the following:

Corollary: With dimension $\nu = 1$, let the interaction be that of any exponential model E_ξ with ξ transcendental or of any Dyson model. Let ϕ be any state which satisfies $\phi(\sigma^A) = 0$ for all A such that $A_3 \neq \varnothing$. Then

$$\rho[\alpha_t(\theta)] \xrightarrow[t\to\infty]{} M\rho[\theta]$$

for every local observable θ.

We conclude this section with an example which shows that one cannot expect the models to be so well behaved on all initial states. In particular, we exhibit a state which shows recurrences for all ferromagnetic Ising-type models.

Define $\phi \in \mathfrak{S}$ as $\otimes_{j\in\mathbf{Z}^\nu} \hat{f}_j$, where $\sigma_z^j f_j = f_j$ for $j \neq 0$, and $\sigma_x^0 f_0 = f_0$. Then, in the GIM, we have from (1) that

$$\phi[\alpha_t(\sigma_x^0)] = \phi[\sigma_x^0 \cos(2P_0 t) - \sigma_y^0 \sin(2P_0 t)].$$

But $\hat{f}_0(\sigma_y^0) = 0$, and so the second term on the rhs vanishes. Hence,

$$\phi[\alpha_t(\sigma_x^0)] = \sum_{n=0}^{\infty} \frac{(it)^{2n}}{(2n)!} \phi(P_0^{2n}) \cdot 2^{2n} = \cos(2pt),$$

where

$$p = \sum_{j\in\mathbf{Z}^\nu} \epsilon(|j|).$$

6. RATE OF DECAY TO EQUILIBRIUM

Most of the proofs in previous sections depended on properties of functions of the form

$$f(t) = \prod_{n=1}^{\infty} \cos(a_n t).$$

We now want to comment on the essential connection of this function with our problem. We show, in particular, that f is the Fourier transform of a certain measure μ of physical origin and that investigation of the structure of this measure can give detailed information about the behavior of the system. Before we can discuss this further, we need some definitions and facts.

Let μ be a Borel probability measure, hereafter abbreviated Bpm. We denote Borel sets with Lebesgue measure zero by Z, and countable sets by C. Then μ is called

(1) absolutely continuous if $\mu(Z) = 0$ for all Z,
(2) singular continuous if $\mu(Z) = 1$ for some Z and $\mu(C) = 0$ for all C,
(3) discontinuous if $\mu(C) = 1$ for some C.

An equivalent classification is obtained by using the function $\mu(I_x)$ of $x \in \mathbf{R}$, called the distribution function of μ, where $I_x = \{y \in \mathbf{R} \mid y \leq x\}$. Then μ is

(1) absolutely continuous if and only if $\mu(I_x)$ is an absolutely continuous point function,
(2) singular continuous if and only if $\mu(I_x)$ is continuous and $d\mu(I_x)/dx = 0$ for almost all x,
(3) discontinuous if and only if the range of $\mu(I_x)$ is a countable set.

If μ_1 and μ_2 are Bpm's, the set function defined by

$$\mu_1 * \mu_2(A) = \int_{\mathbf{R}} \mu_1(A - x)\, d\mu_2(x)$$

is a Bpm called the convolution of μ_1 and μ_2. If μ is a Bpm, the point function defined by

$$\hat{\mu}(t) = \int_{\mathbf{R}} e^{itx}\, d\mu(x)$$

is called the Fourier transform of μ. We then have the following connection: If μ_1 and μ_2 are Bpm's, then $\widehat{\mu_1 * \mu_2}(t) = \hat{\mu}_1(t)\hat{\mu}_2(t)$. Furthermore, the Fourier transform is a means of determining continuity properties of Bpm's since[34] if γ is greater than the positive integer p, then

$$\hat{\mu}(t) = O_\pm(|t|^{-\gamma}) \quad \text{implies that} \quad \mu(I_x) \in C^p. \quad (13)$$

Here, we use the notation that a function $g(t)$ satisfies $g(t) = O_\pm[h(t)]$ if there exist positive constants c and d such that $|g(t)| \leq ch(t)$ for all $t > d$. Also, $g \in C^p$ means that g has continuous derivatives through order p.

We now come back to $\rho[\alpha_t(\sigma_x^0)]$. From Proposition 1, we have

$$\alpha_t(\sigma_x^0) = \exp[i\tilde{H}_0 t]\sigma_x^0 \exp[-i\tilde{H}_0 t],$$

where $\tilde{H}_0 = \sigma_z^0 \sum_{k\in Z^\nu} \epsilon(|k|)\sigma_z^k$. It is then easy to obtain

$$\rho[\alpha_t(\sigma_x^0)] = \rho(\sigma_x^0 \exp[i2\tilde{H}_0 t]).$$

Defining μ as the spectral measure of $\Pi_\rho(2\tilde{H}_0)$ corresponding to the form $\rho(\sigma_x^0 \cdot) = (\Pi_\rho(\sigma_x^0)\Phi_\rho, \cdot\, \Phi_\rho)$, we have from Stone's theorem that

$$\rho(\alpha_t[\sigma_x^0]) = \hat{\mu}(t). \tag{14}$$

By a simple generalization of (13) to complex measures, we have

Proposition 6: In the GIM, if $\rho[\alpha_{|t|}(\sigma_x^0)] = O_\pm(|t|^{-\gamma})$ for any γ greater than the positive integer p, then $\mu(I_x) \in C^p$, where μ is the spectral measure of $\Pi_\rho(\tilde{H}_0)$ corresponding to the form $\rho(\sigma_x^0 \cdot)$.

To show that the available range for the rates of decay into equilibrium is wide enough to be of interest, we investigate the situation discussed in Sec. 5. Therefore, restricting ourselves to dimension $\nu = 1$ and states ρ which satisfy $\rho(\sigma^A) = 0$ for all A such that $A_3 \neq 0$, we have, as in (6′),

$$\rho[\alpha_t(\sigma_x^0)] = \rho(\sigma_x^0)\rho(e^{-i2\tilde{H}_0 t}) = \rho(\sigma_x^0)\prod_{n=1}^{\infty}\cos^2[2\epsilon(n)t].$$

We therefore need to classify the measures μ_L, μ_ξ, and μ_α which have Fourier transform

$$\prod_{n=1}^{\infty}\cos[2\epsilon(n)t],$$

with ϵ coming from the finite-range, exponential, and Dyson models, respectively:

(A) For the finite-range models, it is easy to see that μ_L is discontinuous, producing recurrent behavior.

(B) For the exponential model E_ξ with $\xi > 2$, it is known[35] that μ_ξ is singular continuous so that $\hat{\mu}_\xi \notin C^1$ and, therefore, from (13) that

$$\prod_{n=1}^{\infty}\cos^2\left(\frac{2t}{\xi^n}\right) \neq O_\pm(|t|^{-\gamma}), \quad \text{for any} \quad \gamma > 2 \quad \text{if} \quad \xi > 2.$$

In fact, it is further known[36] that $\mu_\xi * \mu_\xi$ is singular continuous for $\xi > 3$ so that

$$\prod_{n=1}^{\infty}\cos^2\left(\frac{2t}{\xi^n}\right) \neq O_\pm(|t|^{-\gamma}), \quad \text{for any} \quad \gamma > 1 \quad \text{if} \quad \xi > 3.$$

(C) For the Dyson model D_α, the following lemma shows that, for some $c > 0$,

$$\prod_{n=1}^{\infty}\cos^2\left(\frac{2t}{n^\alpha}\right) < \exp[-c\,|t|^{1/\alpha}]$$

so that $\mu_\alpha \in C^\infty$.

Lemma 4: If $\alpha > 1$, there exists a $c > 0$ such that

$$\left|\prod_{j=1}^{\infty}\cos\left(\frac{t}{j^\alpha}\right)\right| \leq \exp[-c\,|t|^{1/\alpha}].$$

Proof: For $0 < x < 1$, we have $0 < \cos x < 1 - cx^2$ for some $c > 0$, and $1 - x \leq e^{-x}$. Therefore, for $t > 0$ we have

$$\left|\prod_{j=1}^{\infty}\cos\left(\frac{t}{j^\alpha}\right)\right| \leq \prod_{j>t^{1/\alpha}}\left|\cos\frac{t}{j^\alpha}\right| \leq \prod_{j>t^{1/\alpha}}\left(1 - c\frac{t^2}{j^{2\alpha}}\right) \leq \exp\left(-c\prod_{j>t^{1/\alpha}}\frac{t^2}{j^{2\alpha}}\right).$$

By integral approximation,

$$\sum_{j>t^{1/\alpha}}\frac{t^2}{j^{2\alpha}} \geq t^{1/\alpha}.$$

The transition to negative t then gives the full result. QED

The above classification shows by example how wide a range of rates of decay is attainable. To complete the picture, we note[37] that for no form of interaction in the GIM is there a $c > 0$ such that

$$\prod_{n=1}^{\infty}\cos^2[2\epsilon(n)t] = O_\pm(e^{-c|t|}).$$

7. CONCLUSIONS

The analysis presented in this paper leads to an explicit statement on the relation, in the thermodynamical limit, between the spectrum of the "Hamiltonian" and the time behavior of the expectation values for local observables. In particular, Proposition 6 shows that, for generalized Ising interactions, the degree of continuity of the spectrum of local Hamiltonians, considered in the Hilbert space generated by any initial state, limits the rate at which that initial state can approach equilibrium.

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[1] F. J. Murray and J. von Neumann, Ann. Math. **37,** 116 (1936).
[2] Z. Takeda, Tôhoku Math. J., Ser. 2, 7, 67 (1955).
[3] This framework was introduced in D. Robinson, Commun. Math. Phys. **6,** 151 (1967).
[4] For more general interactions see Theorem 7.6.2 of D. Ruelle, *Statistical Mechanics* (Benjamin, New York, 1969).
[5] W. Magnus, Commun. Pure Appl. Math. **7,** 649 (1954).
[6] G. G. Emch, H. J. F. Knops, and E. J. Verboven, Commun. Math. Phys. **7,** 164 (1968).
[7] The average can be put in the more understandable form of an integral as follows. By the Riesz representation theorem, the means $\{M\}$ on $CB(\mathbf{R})$ are in 1-to-1 correspondence with the probability measures $\{\mu\}$ on the Stone–Čech compactification $\beta\mathbf{R}$ of $\mathbf{R}$, and the correspondence is $Mf = \int_{\beta\mathbf{R}} f(t)\, d(t)$ for all f in $CB(\mathbf{R})$.
[8] A review of the proof shows that a discrete amenable group (e.g., any Abelian group) will give a left invariant mean on the set of all bound functions on the group, and this mean also gives rise to an invariant state. Here, we have the desired continuity anyway.
[9] See Ref. 6.
[10] Since the group is Abelian, we drop "left."
[11] See Appendix 1 of F. P. Greenleaf, *Invariant Means on Topological Groups* (Van Nostrand–Reinhold , New York, 1969). Also note that an ergodic average of the form $\lim 1/T \int_0^T dt$, as $T \to \infty$, does not exist on $CB(\mathbf{R})$.
[12] W. E. Eberlein, Trans. Am. Math. Soc. **67,** 217 (1949). The part of this work which we are using here has recently been generalized to non-Abelian groups. See, for example, Ref. 11, p. 38.
[13] See Ref. 12.
[14] See Ref. 12.
[15] As in Footnote 7, we can give a setting which gives a simple interpretation of our average but also exhibits its uniqueness. Since all the functions of time that we have are in $W(\mathbf{R})$, Theorem 5.3 of K. Deleeuw and I. Glicksberg [Acta Math. **105,** 63 (1961)] implies that the means on $W(\mathbf{R})$ are in one-to-one correspondence with the probability measures on a compactification $\mathbf{R}^\omega$ of $\mathbf{R}$ which is smaller than $\beta\mathbf{R}$. Therefore, given the orbit $\{\alpha_t^* \rho\}$ of an initial state ρ, we are using as our average the unique integral

$$M\rho = \int_{\mathbf{R}^\omega} \alpha_t^* \rho \, d\mu(t),$$

which is independent of the definition of $t = 0$.
[16] Theorem V.3.15 in N. Dunford and J. T. Schwartz, *Linear Operators,* (Interscience, New York, 1957), Part I.
[17] Lemma 11.24 and Prop. 3.4.2 of J. Dixmier, *Les C*-algèbres et leurs représentations* (Gauthier–Villars, Paris, 1964).
[18] The fact that our algebra is simple and antiliminal is proven in Prop. 3.1 of A. Guichardet, Ann. Sci. Ecole Normale Supér., 3rd Ser. **83,** 1 (1966).
[19] H. J. F. Knops, thesis, University of Nijmegen, 1969.
[20] G. G. Emch, H. J. F. Knops and E. J. Verboven, J. Math. Phys. **11,** 1656 (1970).
[21] H. Araki, Commun. Math. Phys. **14,** 120 (1969).
[22] R. V. Kadison, Top.,Supp. 2, **3,** 177 (1965).
[23] An example of what can be used as a substitute when the obvious limit cannot be proven is contained in Sec. 7.3 of Ref. 4.
[24] See Ref. 21.
[25] The essence of this proof is already contained on page 412 of J. L. Lebowitz, Rev. Phys. Chem. **19,** 389 (1968).
[26] D. G. Kelly and S. Sherman, J. Math. Phys. **9,** 466 (1968).
[27] G. G. Emch, J. Math. Phys. **7,** 1198 (1966).
[28] I. J. Lowe and R. E. Norberg, Phys. Rev. **107,** 46 (1957).
[29] See, for example, M. Kac, *Statistical Independence in Probability, Analysis and Number Theory* (Wiley, New York, 1959).
[30] In fact, the product is absolutely convergent so that we may rearrange factors arbitrarily.
[31] Theorem II in R. Salam, *Algebraic Numbers and Fourier Series* (Heath, Boston, 1963).
[32] F. J. Dyson, Comm. Math. Phys. **12,** 91 (1969).
[33] J. von Neumann, Proc. Natl. Acad. Sci.(U.S.)**18,** 70 (1932).
[34] See p. 117 of A. Wintner, *The Fourier Transforms of Probability Distributions* (Edwards Brothers, Ann Arbor, Michigan, 1947).
[35] J. P. Kahane and R. Salem, Colloq. Math. **6,** 193 (1958).
[36] See Ref. 35.
[37] C. Radin, to be published.

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Classical Thermodynamics Simplified

PETER RASTALL
Physics Department, University of British Columbia, Vancouver 8, B.C.

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Classical thermodynamics is developed in a rigorous and quite general form. The approach is similar to Carathéodory's in that entropy and temperature are defined in terms of quantities which are more directly measurable, but Pfaffian forms and quasistatic processes do not appear. The mathematics used is elementary, apart from a small amount of symbolic logic and a very little topology.

1. INTRODUCTION

The second law of thermodynamics is still often stated in the manner of Kelvin: *It is impossible to construct a system that, operating in a cycle, will produce no effect other than the extraction of heat from a reservoir and the performance of work on a mechanical system.* Such formulations have a comfortingly operational sound, but they are unsatisfactory as a basis for a physical theory. Their most serious defect is that they are incomplete. For example, they give no indication of what processes are possible for a physical system. One is forced to rely on intuitive judgements, which makes it impossible to construct a logically sound theory. To make matters worse, the processes required in traditional applications of the second law are often "quasistatic" or "reversible," and can be defined only by subtle limiting procedures.

Carathéodory[1] was the first to attempt an axiomatic formulation of thermodynamics. Although his theory is not completely general, it does apply to a large class of systems. Heat, entropy, and temperature are defined in terms of measurable quantities, and the assumptions of the older theory are made more explicit and simplified. Despite these considerable