

# Existence of global solutions to the Cauchy problem for the inelastic Boltzmann equation with near-vacuum data

Kaniel & Shinbrot approach

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# Outline

- 1 Introduction
  - The inelastic Cauchy problem
  - Problems associated to the inelasticity
- 2 Main Result
  - Spaces of functions bounded by Maxwellians
  - Main result
- 3 Kaniel & Shinbrot approach
- 4 Apriori estimate and proof of the theorem
- 5 Final observations

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# Inelastic Boltzmann equation

A good model for inelastic interactions between molecules in a diluted gas is given by

$$\frac{\partial f}{\partial t} + v \cdot \nabla f = Q(f, f) \text{ in } (0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \quad (1)$$

where  $f(v, x, t)$  represents the density of molecules of the gas traveling with velocity  $v$  at the space-time point  $(x, t)$ .

# Collision Operator

The Collision Operator  $Q(f, g)$  is given by the bilinear form

$$Q(f, g) = \int_{\mathbb{R}^n} \int_{S^{n-1}} \left\{ \frac{1}{e'J} f(v')g(v_*') - f(v)g(v_*) \right\} |u \cdot n| dndv_*. \quad (2)$$

The parameters  $e$  and  $J$  stand for the restitution coefficient and the Jacobian of the collision law respectively.

# Collision law

Two molecules with velocities  $v$  and  $v_*$  interact by the following law that relates the post-collision relative velocity  $u'$  with the pre-collision one  $u$

$$u' \cdot n = -e(|u \cdot n|)(u \cdot n), \text{ with } n \in S^{n-1} \quad (3)$$

where  $u = v - v_*$  is the relative velocity. It turns up that the velocities of the particles are given by

$$v' = v - \frac{1+e}{2}(u \cdot n)n \text{ and } v_*' = v_* + \frac{1+e}{2}(u \cdot n)n. \quad (4)$$

Note that we assumed that the restitution coefficient depends only on the impact velocity  $|u \cdot n|$ .

## Jacobian of the collision law

We agreed to have a restitution coefficient depending just on the impact velocity, i.e  $e : z \mapsto e(z)$  for  $z \in [0, \infty)$ . It is not hard to realize that the Jacobian of transformation (4) is given by

$$J(z) = \left| \frac{\partial v', v'_*}{\partial v, v_*} \right| = e(z) + ze_z(z) = \theta_z(z) \quad (5)$$

where  $\theta(z) = ze(z)$ . Therefore, the Jacobian is fully determined by the restitution coefficient  $e$ .

## Assumptions of the restitution coefficient

- (A1)  $z \rightarrow e(z)$  is absolute continuous from  $[0, +\infty)$  into  $(0, 1]$ . (Regularity)
- (A2) The mapping  $z \rightarrow \theta(z) = ze(z)$  is strictly increasing. (Invertibility)
- (A3) Define for any  $\beta > 0$  the ratio

$$\psi_\beta = \frac{\exp\left(-\frac{\beta}{2} z^2\right)_z}{\exp\left(-\frac{\beta}{2} \theta^2\right)_z}.$$

Then, the following (integrability) condition on  $\psi_\beta$  must hold

$$\phi_\beta(x) = 2\omega_{n-2} \int_0^1 \psi_\beta(xz) \left(1 - z^2\right)^{\frac{n-3}{2}} dz \in L^\infty([0, +\infty)). \quad (6)$$

## Examples of restitution coefficient

The following are examples of commonly used restitution coefficients.

(1)  $e(z) = e_0 \in (0, 1]$ . Constant restitution coefficient.

(2)  $e(z) = \frac{1}{1+az^\gamma}$ . For  $a > 0$  and  $\gamma \in (0, 1]$ . Monotonic decreasing restitution coefficient.

(3)  $e(z) = \begin{cases} 1 & \text{for } z < z_0 \\ e_0(z) & \text{for } z \geq z_0 \end{cases}$ , for some  $z_0 > 0$ .

Continuous restitution coefficient that is elastic for small velocities and inelastic for high velocities.

The function  $e_0(z)$  is usually decreasing that may go to zero as  $z \rightarrow \infty$ .

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# Properties of the collision operator

The collision operator enjoys the standard properties:

(P1) Conservation of mass and momentum

$$\int_{\mathbb{R}^n} Q(f, f)(a + bv) dv = 0 \text{ for any } a, b \in \mathbb{R}.$$

(P2) Loss of energy or cooling

$$\int_{\mathbb{R}^n} Q(f, f)|v|^2 dv \leq 0.$$

# No entropy decay in general

After some computations one gets

$$\int_{\mathbb{R}^n} Q(f, f) \log(f) dv = -\mathcal{D}(f) + \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{S^{n-1}} \left( \frac{1}{J} - 1 \right) ff_* |u \cdot n| dndv_* dv$$

where  $\mathcal{D}(f) \geq 0$ .

This is the main upset in using the Lions-DiPerna compactness proof of existence of renormalize solutions in this case.

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# The spaces $M^{\alpha,\beta}$

A natural space to seek solution for the Cauchy problem is

$$M^{\alpha,\beta} = \left\{ f \in L^1 : |f| \leq c \exp(-\alpha|x|^2) \exp(-\beta|v|^2) \text{ for some } c > 0 \right\} \quad (7)$$

which is a Banach space with norm

$$\|f\|_{\alpha,\beta} := \inf_{c>0} \left\{ |f| \leq c \exp(-\alpha|x|^2) \exp(-\beta|v|^2) \right\}. \quad (8)$$

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# Mild and weak solution

Using previous notation one has

## Theorem

Let  $T > 0$  and  $f_0 \in M^{\alpha,\beta}$  with

$$\|f_0\|_{\alpha,\beta} \leq \frac{1}{4k_{\alpha,\beta}}$$

where  $k_{\alpha,\beta}$  is given by

$$k_{\alpha,\beta} = C_n \frac{\|\phi_\beta\|_{L^\infty}}{\alpha^{1/2} \beta^{n/2}}.$$

Then, the Cauchy problem has a unique mild-weak solution with  $f^\# \in L^\infty(0, T; M_+^{\alpha,\beta}) \cap C(0, T; L^1)$ .

## Building approximations to the solution

Assume that  $l_0, \dots, l_{n-1}, u_0, \dots, u_{n-1}$  are known, then  $l_n(t)$  and  $u_n(t)$  are the mild solutions in  $[0, T]$  of the linear problems

$$\frac{dl_n^\#}{dt}(t) + Q_-^\#(l_n, u_{n-1})(t) = Q_+^\#(l_{n-1}, l_{n-1})(t)$$

$$\frac{du_n^\#}{dt}(t) + Q_-^\#(u_n, l_{n-1})(t) = Q_+^\#(u_{n-1}, u_{n-1})(t) \quad (9)$$

with  $l_n(0) = u_n(0) = f_0$ .

# Beginning condition

In order to start the process fix in  $[0, T]$  an initial function  $u_0^\# \in L^\infty(0, T; M_+^{\alpha, \beta})$  and the condition

$$0 \leq l_0^\#(t) \leq l_1^\#(t) \leq u_1^\#(t) \leq u_0^\#(t) \quad \text{in } 0 \leq t \leq T \quad (10)$$

# $I_n$ increases and $u_n$ decreases

The beginning condition assures that  $\{I_n\}$  is an increasing sequence and  $\{u_n\}$  is a decreasing sequence. The proof of this fact is simple and uses

- Mild solution formula for the linear problem.
- The fact that the negative collision operator can be written as

$$Q_-(f, g) = \left(2 \frac{\omega_{n-2}}{n-1}\right) \int_{\mathbb{R}^n} g(v_*) |u| dv_* f(v)$$

# The sequences converge to a unique solution

Let  $u$  and  $l$  the limits of the sequences  $\{u_n\}$  and  $\{l_n\}$ .

- The final step in the iteration process uses the linear equations (9) to find an estimate on

$$\left\| u^\#(t) - l^\#(t) \right\|_{L^1(\mathbb{R}^{2n})}$$

and concludes that  $u = l$  using Gronwall's lemma.

- The uniqueness follows by Gronwall's lemma repeating the previous argument.

# Control using the Maxwellian norm of $f^\#$

## Lemma

For any  $0 \leq t \leq T$  and  $f^\# \in L^\infty(0, T; M^{\alpha, \beta})$  the following inequality holds

$$\int_0^t |Q^\#(f, f)(\tau)| d\tau \leq k_{\alpha, \beta} \exp(-\alpha|x|^2 - \beta|v|^2) \|f^\#\|_{L^\infty(0, T; M^{\alpha, \beta})}^2 \quad (11)$$

where

$$k_{\alpha, \beta} = C_n \frac{\|\phi_\beta\|_{L^\infty}}{\alpha^{1/2} \beta^{n/2}}.$$

In other words,

$$\int_0^t |Q^\#(f, f)(\tau)| d\tau \in L^\infty(0, T; M^{\alpha, \beta}) \quad (12)$$

# Key steps in the proof of lemma

The two main points are

- The "independence of time" estimate

$$\int_0^t \exp\left(-\alpha |x + \tau(v - v)|^2\right) \exp\left(-\alpha |x + \tau(v - v_*)|^2\right) d\tau \leq \frac{\sqrt{\pi}}{\alpha^{1/2}|u|} \exp\left(-\alpha |x|^2\right) \quad (13)$$

- Integrability assumption (A3)

## Proof of the main theorem

One only needs to satisfy the beginning condition globally in time.

- This is assured by choosing

$$l_0^\# = 0 \quad \text{and} \quad u_0^\# = C \exp\left(-\alpha|x|^2 - \beta|\xi|^2\right)$$

with  $C$  sufficiently small.

- Using the explicit equation for mild solutions it is possible find the conditions on  $f_0$

$$\|f_0\|_{\alpha,\beta} \leq \frac{1}{4k_{\alpha,\beta}}$$

# Observations

- Inelasticity does not play a big role for dilute gases in the free space. In fact, the equilibrium is always the trivial state. Note that the theorem implies that

$$f \leq C \exp(-\alpha|x - t\xi|^2) \exp(-\beta|\xi|^2) \longrightarrow 0 \text{ as } t \rightarrow \infty$$

- A more difficult question arise for bounded domains with inelastic reflections. Recall, that Shinbrot and Kaniel technique was first applied for this case. In the inelastic setting is not so easy to use. Perhaps, this is due to a possible clustering near the boundary.

# Bibliography



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