Propagation of L^1 and L^∞ Maxwellian weighted bounds for derivatives of solutions to the homogeneous elastic Boltzmann Equation

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Outline

1. Introduction
   - Problem and assumptions
   - Collision Operator (CO)
2. Results
   - Known Results
   - Main Results
3. Key ingredients of the proof
   - Moments of the CO derivatives
   - System of ODE's
   - Induction in the multi-index order
   - $L^\infty - L^1$ CO comparison

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Homogeneous Boltzmann Equation (HBE)

The HBE is given by

$$\frac{\partial f}{\partial t} = Q(f,f) \ \text{ in } \ (0,+\infty) \times \mathbb{R}^n \tag{1}$$

where $f(v,t)$ represents the density of molecules of the gas traveling with velocity v at the time t.

Collision Operator

The Collision Operator $Q(f,g)$ is given by the bilinear form

$$Q(f,g) = \int_{\mathbb{R}^n} \int_{S^{n-1}} \left\{ f('v)g('v_*) - f(v)g(v_*) \right\} B(v - v_*, \sigma) d\sigma dv_*. \tag{2}$$

Collision law

Two molecules with velocities v and v_* interact by the following law that relates the post-collisional velocities with the pre-collisional ones

$$v' = v + \frac{1}{2}(|u|\sigma - u)), \quad v'_* = v_* - \frac{1}{2}(|u|\sigma - u)) \tag{3}$$

where $u = v - v_*$ is the relative velocity.

Collision kernel

We use the variable hard potential collision kernel

$$B(v - v_*, \sigma) = |v - v_*|^{\alpha}\, h(\hat{u} \cdot \sigma) \;\text{ and }\; \hat{u} = \frac{v - v_*}{|v - v_*|}$$

with $\alpha \in (0, 1]$ and $\hat{u}$ is the renormalized relative velocity. It is assumed that the angular cross section $h(\cdot)$ has the following properties

(i) $h(z) \geq 0$ is nonnegative on $(-1, 1)$ such that

$$h(z) + h(-z) \text{ is nondecreasing on } (0, 1)$$

(ii)

$$0 \leq h(z)(1 - z^2)^{\mu/2} \leq C \text{ for } z \in (-1, 1)$$

where $\mu < n - 1$ and $C > 0$ constant.

Last property implies the *Grad cutoff assumption*.

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Properties of the Collision Operator

The collision operator enjoys the properties:

(P1) Conservation of mass, momentum and energy

$$\int_{\mathbb{R}^n} Q(f,f)(a+b\,v+c\,|v|^2)dv = 0 \text{ for any } a,b,c \in \mathbb{R}.$$

(P2) Differentiation formula. For any multi-index η

$$\partial^\eta Q(f,f) = \sum_{\nu \le \eta} \binom{\eta}{\nu} Q(\partial^\nu f, \partial^{\eta-\nu} f).$$

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Regularity

Assume that $h(\cdot)$ has the properties discussed. Then for a solution of the HBE

(1) If f_0 satisfies $\int_{\mathbb{R}^n} f_0 \exp(r_0|v|^2)dv < \infty$ for some $r_0 > 0$, then there exist $r \leq r_0$ such that $\sup_{t \geq 0} \int_{\mathbb{R}^n} f \exp(r|v|^2)dv < \infty$.

(2) If $f_0 \leq K_0 \exp(-r_0|v|^2)$ for some $K_0, r_0 > 0$ then there exist $r \leq r_0$ such that $f \leq K \exp(-r|v|^2)$ for all $t \geq 0$ and some positive constants K.

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L^1 Maxwellian propagation

Let η any multi-index and assume that

$$f_0 \in W^{|\eta|,1}_{2+\alpha} \cap H^{|\eta|}_{(|\eta|-1)(1+\alpha/2)}.$$

In addition, assume that for any $\nu \leq \eta$ one has

$$\int_{\mathbb{R}^n} |\partial^\nu f_0| \exp(r_0|v|^2) dv < \infty$$

for some $r_0 > 0$. Then

$$\sup_{t\geq 0} \int_{\mathbb{R}^n} |\partial^\nu f| \exp(r|v|^2) dv < \infty$$

for all $\nu \leq \eta$ and some $r \leq r_0$.

L^∞ Maxwellian propagation

Define $M_r := \exp(-r|v|^2)$. Let η any multi-index and assume that

$$f_0 \in H^{|\eta|}_{(|\eta|-1)(1+\alpha/2)}.$$

In addition, assume that

$$|\partial^\nu f_0| / \left\{ (1+|v|^2)^{|\nu|/2} M_{r_0} \right\} \in L^\infty$$

for some $r_0 > 0$. Then, there exist $r \leq r_0$ such that

$$\sup_{t \geq 0} \frac{|\partial^\nu f|}{(1+|v|^2)^{|\nu|/2} M_r} \leq K_{\eta, r_0}$$

for all $\nu \leq \eta$, where K_{η, r_0} is a positive constant depending on η, r_0 and the kernel $h(\cdot)$.

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Control of the moments of the CO derivatives

Assume $h(z)$ fulfills all the conditions discussed, then for every $p > 1$ and multi-index η

$$\int_{\mathbb{R}^n} \partial^\eta Q(f,f) sgn(\partial^\eta f)|v|^{2p} dv \leq -(1-\gamma_p)k_\alpha \delta^\eta m_{p+\alpha/2} + \gamma_p \delta^\eta S_p + \delta^{\eta^-}(m_{\alpha/2} m_p) + \delta^{\eta^-}(m_0 m_{p+\alpha/2}) \quad (4)$$

where k_α is a positive constant depending on α but not on p, and

$$\delta^\eta m_p := \int_{\mathbb{R}^n} |\partial^\eta f|\, |v|^{2p} dv \;\; and \;\; \gamma_p \sim 1/p^\epsilon$$

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System of ODE's for the normalized moments $\delta^\eta z_p$

The normalized moments

$$\delta^\eta z_p := \frac{\delta^\eta m_p}{\Gamma(p+b)}$$

satisfy the following infinite system of ODE's.

$$\frac{d(\delta^\eta z_p)}{dt} + (1-\gamma_p)k_0\Gamma(p+b)^{\alpha/2p}\delta^\eta m_0^{-\alpha/2p}(\delta^\eta z_p)^{1+\alpha/2p}$$
$$\leq \gamma_p k_1 p^{\alpha/2+b}\,\delta^\eta Z_p + k_2 p^{\alpha/2}\delta^{\eta^-}(m_0 z_{p+\alpha/2}) + \delta^{\eta^-}(m_{\alpha/2} z_p) \quad (5)$$

for all $p > 1$, with k_0, $k_1 > 0$ and k_2 universal positive constants. The term $\delta^\eta Z_p$ only depends on derivative moments $q \leq p$ and $\nu < \eta$.

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Induction using the system of ODE's

- Since the term $\delta^\eta Z_p$ only depends on moments of previous derivatives one can use an induction argument in the multi-index order on the system (5). The first step of the induction is given by the known result on the moments for the solution f.
- It follows that the result on the derivatives of f is as strong as the first step of the induction, i.e. is as strong as the result on f.

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For the L^∞ result

Once the L^1 Maxwellian propagation is proved, it is possible to use the following comparison principle in the CO (Gamba, Panferov & Villani).

Theorem

Assume $B(u,\sigma) = |u|^\alpha h(\hat{u}\cdot\sigma)$ with $h(\cdot)$ satisfying the conditions stated in the introduction. Then for any measurable function $g \geq 0$,

$$\left\| \frac{Q^+(g, M_r)}{M_r} \right\|_{L^\infty} \leq K \left\| \frac{g}{M_r} \right\|_{L^1}$$

for some positive constant K depending on α and r.

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