

# On the diffusive wave approximation of the Shallow Water equations

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## Abstract

In this paper we study basic properties of the diffusive wave approximation of the Shallow Water equations (DSW). This equation is a doubly nonlinear and degenerate diffusion equation arising in shallow water flow models. It has been used as a model to simulate water flow driven mainly by gravitational forces and dominated by shear stress, that is, under uniform and fully developed turbulent flow conditions. These flow conditions occur for example in marshes, wetlands, and overland flow in vegetated areas. The aim of this work is twofold. On one hand, we intend to provide the engineering community with a survey of relevant results coming from the studies of doubly nonlinear diffusion equations that can be applied to the DSW equation when topographic effects are ignored. In fact, we present proofs of the most relevant results existing in the literature using constructive techniques that directly lead to the implementation of numerical algorithms to obtain approximate solutions. On the other hand, we want to introduce to both, the engineering and mathematical communities, the problem that arises when topographic effects are considered (obstacle problem). This is to the best of our knowledge a new avenue of research in the area of theoretical PDEs.

**Keywords:** Doubly nonlinear, quasilinear elliptic-parabolic, degenerate parabolic, shallow water equations, doubly degenerate diffusion.

## 1 Introduction

In this paper we study basic properties of the diffusive wave approximation of the Shallow Water equations (DSW). This equation is a doubly nonlinear and degenerate diffusion equation arising in shallow water flow models. It has been used as a model to simulate water flow driven mainly by gravitational forces and dominated by shear stress, that is, under uniform and fully developed turbulent flow conditions. These flow conditions occur for example in marshes, wetlands, and overland flow in vegetated areas, see [18], [10], [19], [8]. The purpose of this paper is twofold. On one hand, we intend to provide the engineering community with a survey of relevant results coming from the studies of doubly nonlinear diffusion equations that can be applied to the DSW equation when topographic effects are ignored. In fact, we present proofs of the most relevant results existing in the literature using constructive techniques that directly lead to the implementation of numerical algorithms to obtain approximate solutions. On the other hand, we want to introduce to both,

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the engineering and mathematical communities, the problem that arises when topographic effects are considered (obstacle problem).

The DSW equation gives rise to the following initial/boundary-value problem prescribed for any fixed  $T > 0$

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} - \nabla \cdot \left( (u - z)^\alpha \frac{\nabla u}{|\nabla u|^{1-\gamma}} \right) = f & \text{on } \Omega \times (0, T] \\ u = u_0 & \text{on } \Omega \times \{t = 0\} \\ \left( \frac{(u - z)^\alpha}{|\nabla u|^{1-\gamma}} \nabla u \right) \cdot n = g_N & \text{on } \partial\Omega \cap \Gamma_N \times (0, T] \\ u = g_D & \text{on } \partial\Omega \cap \Gamma_D \times (0, T] \end{array} \right. \quad (1)$$

where  $\Omega$  is an open, bounded subset of  $\mathbb{R}^n$  ( $n = 1, 2$ ) and  $\Gamma_N$  and  $\Gamma_D$  are subsets of  $\partial\Omega \in C^1$  such that  $\partial\Omega = \Gamma_N + \Gamma_D$ . Also  $f : \Omega \times (0, T] \rightarrow \mathbb{R}$ ,  $u_0 : \Omega \rightarrow \mathbb{R}$ ,  $g_N : \Gamma_N \times (0, T] \rightarrow \mathbb{R}$ ,  $g_D : \Gamma_D \times (0, T] \rightarrow \mathbb{R}$  are given functions,  $z : \bar{\Omega} \rightarrow \mathbb{R}^+$  is a positive time independent function,  $0 < \gamma \leq 1$ ,  $1 < \alpha < 2$ , and  $u : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}$  is the unknown. Here  $|\cdot| : \mathbb{R}^n \rightarrow \mathbb{R}$  refers to the Euclidean norm in  $\mathbb{R}^n$ .

Problem (1) is characterized as doubly nonlinear since the nonlinear behaviour appears inside the divergence term as a product of two nonlinearities involving  $u - z$  and  $\nabla u$ , namely  $(u - z)^\alpha$  and  $\nabla u / |\nabla u|^{1-\gamma}$ . In the context of shallow water modeling,  $u(x, t)$  represents the surface water elevation in the position  $x$  at time  $t$ , the positive time independent function  $z(x)$  describes the bathymetry of the bed surface throughout the domain and introduces the commonly called *topographic effects* into the model. In order for equation (1) to serve as a suitable model to simulate water flow, two requirements are needed, the first one being that the water depth be nonnegative,  $u - z \geq 0$ , and the second one being that the gradient of the water elevation,  $\nabla u$ , be comparable to the gradient of the bathymetry  $\nabla z$ , which is usually small. The latter requirement characterizes water flow regimes not far from uniform flow conditions in open channels. The types of physical boundary conditions appropriate for this model are two, a prescribed water depth  $g_D$  on  $\Gamma_D$ , and/or a prescribed water flux  $g_N$  on  $\Gamma_N$ . The first one corresponds to a Dirichlet type boundary condition and it is mostly used to model an infinite source of water on the boundary  $\Gamma_D$ . The second one corresponds to a Newman type boundary condition and it is the most natural choice to model water flux through a boundary  $\Gamma_N$ .

The outline of the paper is the following. We begin by providing a brief derivation of problem (1) and discussing its relevance in the context of shallow water flow modeling. We then proceed to present the most relevant results in studies of doubly nonlinear diffusion equations existing in the literature that can be applied to the DSW equation when topographic effects are ignored. We present a simple, concise, and constructive proof of existence of solutions to the zero-Dirichlet initial/boundary value problem (4) using the Faedo Galerkin method, which lends itself as a natural numerical algorithm to find approximate solutions. Most of the techniques presented in this proof were originally introduced by Lions [12] and further developed for quasilinear and doubly nonlinear parabolic equations by Alt and Luckhaus [1] and Bernis[4], respectively. We continue the presentation by showing proofs of basic regularity results using *a priori* estimates. In the subsequent section we address the

issues of comparison of solutions first introduced in the context of doubly nonlinear equation by Bamberger [2], and use this result to find nonnegativity and uniqueness of solutions.

## 1.1 Motivation

Models for surface water flows are derived from the incompressible, three-dimensional Navier-Stokes equations, which consist of momentum equations for the three velocity components and a continuity equation. Depending on the physics of the flow, scaling arguments are used in order to obtain effective equations for the problem at hand. See [16]. Equation (1) is a simplified version of the two-dimensional shallow water equations called the diffusive wave or zero-inertia approach, which neglects the inertial terms in the horizontal momentum equations.

Recall that in shallow water theory, the main scaling assumption is that the vertical scales are small relative to the horizontal ones. This approximation reduces the vertical momentum equation to the hydrostatic pressure relation

$$\frac{\partial p}{\partial y} = \rho g$$

where  $g$  is the gravitational constant,  $y$  the vertical coordinate and  $p$  the pressure, and leaves us with two effective momentum equations in the horizontal direction. Upon vertical integration, we can obtain the 2-D shallow water momentum equations. In the diffusive wave approximation, the depth averaged horizontal momentum equations are further approximated using empirical laws, such as Manning's formula or Chézy's formula, to find an effective expression for the horizontal velocity of the fluid in terms of the free water surface slope, given by

$$V = -\frac{(H-z)^{\alpha-1}}{c_f} \frac{\nabla H}{|\nabla H|^{1-\gamma}}, \quad (2)$$

where  $H(t, x)$  is the free water surface elevation or hydraulic head,  $z(x)$  is the bed surface, bathymetry, or land elevation,  $0 < \gamma \leq 1$  and  $1 < \alpha < 2$  are non-negative parameters, and  $c_f(x)$  is a friction coefficient. The resulting effective model is given by a doubly nonlinear and degenerate parabolic equation for the water elevation  $H$ , obtained from substituting the particular form of the depth averaged horizontal velocity given by (2), into an equation that arises from combining the depth averaged continuity equation with the free surface boundary condition. Equation (1) is a simplification of such a model. Furthermore, it is more commonly found in the literature written as

$$\frac{\partial H}{\partial t} - \nabla \cdot \left( \frac{(H-z)^\alpha}{c_f} \frac{\nabla H}{|\nabla H|^{1-\gamma}} \right) = f(t, x), \quad \text{for } (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \quad (3)$$

where  $f(t, x)$  is a source/sink (such as rain or infiltration). Equation (3) has proved to be suitable to model shallow water flow under uniform flow conditions, *i.e.* when the fluid motion is dominated by gravity and balanced by the boundary shear stress. It has been used to simulate, for example, overland flow and flow in wetlands. See [18], [10], [19], [8].

**Remark 1.1.** Note that if one identifies the water elevation  $H$  with the hydrostatic pressure  $p$ , the expression that relates the velocity and the water elevation gradient (2) becomes a nonlinear version of the empirical *Darcy's law* for gas flow through a porous medium. Indeed, flow in vegetated areas such as wetlands can be understood as a flow through a porous medium.

**Remark 1.2.** In this context, equation (3) makes sense physically only if  $H - z \geq 0$ . Note that in writing (1) we have assumed that  $z \equiv 0$  and  $c_f(x) \equiv 1$ . In our study we will be interested in finding nonnegative solutions of problem (1).

**Remark 1.3.** Whenever  $H - z = 0$  (or alternatively  $u = 0$  in (1)), equation (3) degenerates, *i.e.* it is no longer of parabolic type.

**Remark 1.4.** Note in particular that for the case when  $\gamma = 1$ ,  $c_f \equiv 1$ , and  $z \equiv 0$ , equation (3) becomes the Porous Medium Equation (PME). One should expect similarities between the PME and the more general equation (3), although some differences may arise. See section 1.2. A comprehensive study of the PME can be found in the book by Vázquez [15].

## 1.2 Literature Review

To the best of our knowledge, the DSW equation has not been studied in its general form (1). However, when topographic effects are neglected ( $z \equiv 0$ ) and zero-Dirichlet initial/boundary conditions are assumed ( $\partial\Omega = \Gamma_D$ ), one can find a fairly extensive number of works that study doubly nonlinear equations that are relevant to the DSW equation. See for example [12], [13],[9], [2], [11]. Most of these works study alternative formulations of problem (1). These will be explained in the subsequent sections. In this paper we will focus our attention on the alternative formulation given by

$$\begin{cases} \frac{\partial\phi(v)}{\partial t} - \eta^\gamma \nabla \cdot \left( \frac{\nabla v}{|\nabla v|^{1-\gamma}} \right) = f & \text{on } \Omega \times (0, T] \\ v = 0 & \text{on } \partial\Omega \times [0, T] \\ v = v_0 & \text{on } \Omega \times \{t = 0\} \end{cases} \quad (4)$$

where  $\Omega$  is either  $\mathbb{R}^n$  or an open (and in most cases bounded) subset of  $\mathbb{R}^n$ ,  $\eta$  is a positive constant, and the function  $\phi(s) \in C^{0,\eta}(\mathbb{R})$  is an *odd* function satisfying the following properties:

- (i)  $|\phi(s)| \leq |s|^\eta$  for  $0 < \eta \leq \gamma < 1$ , with equality for  $|s| \geq R$  for some  $R \geq 0$
- (ii)  $\phi(s)$  is a concave increasing function for  $s \geq 0$ .

Note that with the change of variables defined by  $u = \phi(v)$ , problem (4) is transformed into

$$\begin{cases} \frac{\partial u}{\partial t} - \eta^\gamma \nabla \cdot \left( ((\phi^{-1})'(u))^\gamma \frac{\nabla u}{|\nabla u|^{1-\gamma}} \right) = f & \text{on } \Omega \times (0, T] \\ u = 0 & \text{on } \partial\Omega \times [0, T] \\ u = u_0 & \text{on } \Omega \times \{t = 0\} \end{cases} \quad (5)$$

Now, choosing

$$0 < \eta = \frac{\gamma}{\alpha + \gamma} < 1, \quad \text{and} \quad \phi(s) = \frac{s}{|s|^{1-\eta}} \quad (6)$$

we can obtain the explicit expression for

$$(\phi^{-1})'(s) = (1 + \theta)|s|^\theta \quad \text{where} \quad \theta = \frac{1 - \eta}{\eta} = \frac{\alpha}{\gamma} \quad (7)$$

which yields the following equation

$$\frac{\partial u}{\partial t} - \nabla \cdot \left( |u|^\alpha \frac{\nabla u}{|\nabla u|^{1-\gamma}} \right) = f. \quad (8)$$

The previous manipulations lead us to conclude that, at least formally, nonnegative solutions of problem (4) are solutions of the original problem (1) under the aforementioned assumptions.

### 1.2.1 Existence of solutions

Lions [12] introduced the techniques of compactness and monotonicity later utilized in the subsequent works in the proofs of existence for problem (4). Raviart [13], and Grange and Mignot [9] prove the existence of weak solutions to problem (4), provided  $\Omega$  is an open and bounded subset of  $\mathbb{R}^n$ , constructing approximate solutions using implicit finite differences schemes in time and passing to the limit by means of compactness and monotonicity. In [13] the author worked directly with problem (4), and in [9] the authors extended such results to the abstract setting of equations of the type:

$$\frac{\partial Bu}{\partial t} + Au = f$$

where A and B denote the subdifferentials of convex functionals. Their analysis is based on the essential restriction that these functionals must be continuous on appropriate Banach spaces. Bernis further extends these results to the case when  $\Omega$  is any open set of  $\mathbb{R}^n$  in [4]. Another relevant reference is the work of Blanchard and Francfort [5]. In their study the authors address the semi-abstract problem

$$\frac{\partial}{\partial t} b(u) - \nabla \cdot (D\Phi(\nabla u)) = f$$

where  $b$  is a locally Lipschitz function and may grow faster than any power function at infinity, and  $\Phi$  is a  $C^1$  convex functional with specific coercivity assumptions. They obtain existence and comparison results with the aid of a Galerkin approximation technique which uses truncation-penalization of the time nonlinearity and *a priori* estimates through convex conjugate functions.

### 1.2.2 Comparison principles and Uniqueness

In [2], Bamberger studies the existence of particular solutions to problem (4) which are the limit of *solutions fortes* *i.e.* solutions that have the property  $\phi(u)_t \in L^1(0, T, L^1(\Omega))$ . The author refers to this kind of solutions as *limite de solutions fortes*. In addition, the author presents a very concise exposition of a comparison principle between solutions that are *limite de solutions fortes* and uses this result to find uniqueness. See section 4.

### 1.2.3 Regularity

When topographic effects are neglected ( $z \equiv 0$ ) and zero-Dirichlet initial/boundary conditions are assumed ( $\partial\Omega = \Gamma_D$ ) Problem (1) can also be re-written in the form:

$$\frac{\partial u}{\partial t} - \nabla \cdot (|\nabla u^m|^{\gamma-1} \nabla u^m) = f$$

with  $m = 1 + \alpha/\gamma$ . Esteban and Vázquez [6] studied this equation in 1-D for the Cauchy problem ( $\Omega = \mathbb{R}$ ). They study the *local velocity of propagation*

$$V(x, t) = -v_x |v_x|^{\gamma-1}$$

where  $v$  is the nonlinear potential defined as:

$$v = \begin{cases} \frac{m\gamma}{m\gamma - 1} u^{\frac{m\gamma-1}{\gamma}} & \text{if } m\gamma \neq 1 \\ \frac{1}{\gamma} \log u & \text{if } m\gamma = 1 \end{cases}$$

Recall that in the DSW equation, it is assumed that  $m\gamma = \alpha + \gamma > 1$ . In their work, they base their approach on the existing theory for the Porous Medium Equation and find the estimate

$$V_x \leq \frac{1}{\gamma(m+1)t}.$$

Using the previous estimate as the main tool, they construct a theory for the Cauchy problem with nonnegative, integrable initial data. In particular, they address the following questions:

- Existence, uniqueness and regularity of strong solutions,
- Existence and regularity of free boundaries,
- Asymptotic behaviour of solutions and free boundaries.

In [11], Ishige gives a sufficient condition for the growth order of the initial data at infinity for the existence of weak solutions of the Cauchy problem ( $\Omega = \mathbb{R}$ ) (4).

#### 1.2.4 Additional properties of solutions

Some interesting facts about nonnegative solutions to problem (4) are:

- *Finite speed of propagation.* Indeed, Barenblatt constructed a class of self-similar source type solutions for the Cauchy problem ( $\Omega = \mathbb{R}$ ) which have the property that their supports propagate in time with finite speed, when  $\alpha + \gamma > 1$ . See [3].
- *Extinction property.* In [2], using simple arguments, Bamberger exhibits that for  $f = 0$ , nonnegative solutions to the zero-Dirichlet boundary value problem ( $\Omega \subset \mathbb{R}$ , bounded) become zero in finite time.
- *Traveling waves.* It is worthwhile mentioning that an interesting example of traveling wave type solutions

$$u(x, t) = U(t - n \cdot x) \quad \text{with } U(s) = 0 \quad \text{for } s > 0,$$

to the zero-Dirichlet boundary value problem ( $\Omega \subset \mathbb{R}$ , bounded) is shown in [2] for the case when  $\eta > \gamma$  (equivalently  $\alpha < 1 - \gamma$ ). In the DSW equation this case does not arise since  $\alpha > 1$  and  $0 < \gamma \leq 1$ .

Other properties of solutions including *nonexistence of global nonnegative solutions* and *blow up solutions* can be found in [4] and [11] respectively, for particular choices of the parameters  $\eta$  and  $\gamma$  that do not happen in the DSW equation case.

### 1.2.5 About this paper

In this paper we present a simple and constructive proof of existence of solutions. This constructive method provides a natural setting for a computational method to find approximate solutions to problem (4), further described in [14], within the framework of finite element techniques using piecewise polynomial basis functions. Instead of following the time discretization approach established in [13] and [9] we take advantage the continuous in time evolution of the appropriate Banach space norms of the approximate solutions and find *a priori* estimates for them. This is a standard technique proposed in [12] that does not require any truncation-penalization technique as the one used in [5]. Our approximate solutions are *solutions fortes* in the sense of Bamberger [2], and thus, they and their limit will satisfy all the results presented in [2]. In particular, his result on uniqueness of *limite de solutions fortes* will ensure that the numerical scheme will converge to a unique solution. See section 4. For a more detailed study on error estimates and convergence analysis of approximate solutions using the continuous Galerkin method see [14]. It is important to note that in our study we do not require the nonlinearity in time to be locally Lipschitz as in [5].

In addition, we include a concise argument to prove the  $L^\infty$  control and integrability properties of the time derivative of solutions. Although these results have been studied, the regularity arguments we present are hard to find in the literature and provide insight on the complexities of the equation.

For completeness, we include the proof of a comparison result mentioned in [2], and use it to prove uniqueness, nonnegativity and stability of the proposed approximation scheme. These findings are then related to problem (5) through corollaries and observations.

### 1.3 Topographic effects

To the best of our knowledge, existence, uniqueness, and regularity of solutions of the DSW equation in its general form (1), *i.e.* when *topographic* effects are considered, have not been studied. Observe that when one formally carries out the spatial differentiation inside the divergence term in the first equation,

$$\frac{\partial u}{\partial t} - h_1(u, z) \nabla(u - z) \cdot \nabla u - h_2(u, z) \nabla \cdot \left( \frac{\nabla u}{|\nabla u|^{1-\gamma}} \right) = f,$$

where  $h_1(u, z) = \alpha(u - z)^{\alpha-1} / |\nabla u|^{1-\gamma}$  and  $h_2(u, z) = (u - z)^\alpha$ , one can see the appearance of a nonlinear advection term, and a nonlinear diffusive term involving the bathymetry. The topographic effects change qualitatively the direction of the advection  $\nabla(u - z)$ , and scale both, the advection and the diffusion terms. In fact, from the expressions of  $h_1$  and  $h_2$  one can see that for small values of  $u - z$ , the advection term becomes dominant. Some of the difficulties that arise when one introduces a non flat bathymetry  $z$  are:

- The aforementioned techniques to prove existence of solutions (used when  $z = 0$ ) fail, since one cannot send the nonlinearity  $(u - z)^\alpha$  to the time derivative term. In other words the change of variables described at the beginning of section 1.2 does not work correctly. This situation introduces further difficulties when trying to prove the validity of the Galerkin method as a suitable way to obtain approximate solutions.
- In general one expects the regularity of solutions of problem (1) to depend on the properties of  $z$ . Technically speaking, it is not clear how to proceed in order to

incorporate such properties in the analysis and relate them directly with the properties of  $u$ .

#### 1.4 Notation

We will use the standard notation introduced in [7]. Let  $X$  be a real Banach space, with norm  $\|\cdot\|$ . The symbol  $L^p(0, T; X)$  will denote the Banach space of all measurable functions  $u : [0, T] \rightarrow X$  such that

$$(i) \quad \|u\|_{L^p(0, T; X)} := \left( \int_0^T \|u(t)\|^p \right)^{1/p} < \infty, \quad \text{for } 1 \leq p < \infty, \text{ and}$$

$$(ii) \quad \|u\|_{L^\infty(0, T; X)} := \text{ess sup}_{0 \leq t \leq T} \|u(t)\| < \infty.$$

We will denote with  $C([0, T]; X)$  the space of all continuous functions  $u : [0, T] \rightarrow X$  such that

$$\|u\|_{C(0, T; X)} := \max_{0 \leq t \leq T} \|u(t)\| < \infty.$$

Let  $u \in L^1(0, T; X)$ , we say  $v \in L^1(0, T; X)$  is the weak time derivative of  $u$ , denoted  $u_t = v$ , provided

$$\int_0^T \psi_t(t) u(t) = - \int_0^T \psi(t) v(t)$$

for all scalar test functions  $\psi \in C_0^\infty(0, T)$ . Throughout the paper,  $W^{1,p}(0, T; X)$  will denote the space of all functions  $u \in L^p(0, T; X)$  such that  $u_t$  exists in the weak sense and  $u_t \in L^p(0, T; X)$  with the norm

$$\|u\|_{W^{1,p}(0, T; X)} := \begin{cases} \left( \int_0^T \|u(t)\|^p + \|u_t(t)\|^p \right)^{1/p} & (1 \leq p < \infty), \\ \text{ess sup}_{0 \leq t \leq T} (\|u(t)\| + \|u_t(t)\|) & (p = \infty). \end{cases}$$

For  $1 \leq p \leq +\infty$ , we will denote its conjugate as  $p^*$  *i.e.*,  $1/p + 1/p^* = 1$ . For any measurable set  $E \subset \Omega$  and real valued vector functions  $u \in L^p(E)$  and  $v \in L^{p^*}(E)$  we will denote the duality pairing between  $u$  and  $v$  as

$$(u, v)_E := \int_E u \cdot v.$$

For simplicity, we use  $(u, v) := (u, v)_\Omega$ . Similarly, we will denote the duality pairing between  $u \in W^{-1,p^*}(\Omega)$  and  $v \in W_0^{1,p}(\Omega)$  as  $\langle u, v \rangle$ . Recall that the elements of  $W^{-1,p^*}(\Omega)$  are the distributions that have continuous extension to  $W_0^{1,p}(\Omega)$ . These spaces are characterized in the following way: if  $u \in W^{-1,p^*}(\Omega)$ , then there exists functions  $f^0, f^1, \dots, f^n$  in  $L^{p^*}(\Omega)$  such that

$$\langle u, v \rangle = (f^0, v) + \sum_{i=1}^n (f^i, v_{x_i}).$$

Throughout the paper,  $C$  will be a generic constant with different values and the explicit dependence with respect to parameters will be written inside parenthesis.

## 1.5 Definitions of Weak Solution

**Definition 1.1.** We say a function

$$v \in L^{1+\gamma}(0, T; W_0^{1, (1+\gamma)}(\Omega)), \quad \text{with } \phi(v)_t \in L^{(1+\gamma)^*}(0, T; W^{-1, (1+\gamma)^*}(\Omega)),$$

is a weak solution of the initial/boundary-value problem (4) provided

$$\langle \phi(v)_t, w \rangle + \eta^\gamma \left( \frac{\nabla v}{|\nabla v|^{1-\gamma}}, \nabla w \right) = (f, w) \quad \text{a.e in time } 0 \leq t \leq T, \quad (9)$$

for any  $w \in W_0^{1, (1+\gamma)}(\Omega)$  and

$$v(0) = v_0. \quad (10)$$

**Definition 1.2.** We say a function  $u$ , with the properties

$$\phi^{-1}(u) \in L^{1+\gamma}(0, T; W_0^{1, 1+\gamma}(\Omega)), \quad \text{and } u_t \in L^{(1+\gamma)^*}(0, T; W^{-1, (1+\gamma)^*}(\Omega)),$$

is a weak solution of the initial/boundary-value problem (5) provided

$$\langle u_t, w \rangle + \eta^\gamma \left( ((\phi^{-1})'(u))^\gamma \frac{\nabla u}{|\nabla u|^{1-\gamma}}, \nabla w \right) = (f, w) \quad \text{a.e in time } 0 \leq t \leq T, \quad (11)$$

for any  $w \in W_0^{1, (1+\gamma)}(\Omega)$  and

$$u(0) = u_0. \quad (12)$$

**Remark 1.5.** It will be clear from the proof of existence that a consequence of equation (9) (resp. (11)) is that

$$\phi(v) \in C([0, T]; W^{-1, (1+\gamma)^*}) \quad (\text{resp. } u \in C([0, T]; W^{-1, (1+\gamma)^*}))$$

thus condition (10) (resp. (12)) makes sense.

**Remark 1.6.** Observe that in Definition 1.2,  $u$  need not be in any particular Sobolev space. Instead this regularity condition is imposed on  $\phi^{-1}(u)$ . Indeed, for some  $v \in L^{1+\gamma}(0, T; W_0^{1, 1+\gamma}(\Omega))$ , we require that  $u = \phi(v)$ . Since for any  $\phi$  with properties (i) and (ii), the function  $\phi'(s)$  is not necessarily bounded at  $s = 0$ , integrability properties are lost for the distributional gradient of  $u$ . Therefore, it may be that the distribution  $\nabla u$  is not a regular distribution and hence Definition 1.2 would not make sense. The following definition makes precise what we understand for  $\nabla u$ .

**Definition 1.3.** Let  $v \in W^{1, (1+\gamma)}(\Omega)$ , and  $\phi$  with the properties (i) and (ii). Set  $u = \phi(v)$ , then we define the pointwise gradient of  $u$ , denoted as  $\nabla u$ , as the measurable function

$$\nabla u = \begin{cases} \phi'(v)\nabla v & \text{if } |v| > 0 \\ 0 & \text{if } v = 0. \end{cases}$$

## 2 Existence

In order to prove the existence of a weak solution of problem (4) we will use the Faedo-Galerkin method using compactness and monotonicity arguments as explained in [12]. The method consists of five main steps:

*Step 1.* Constructing approximate solutions by the method of Faedo-Galerkin.

*Step 2.* Finding *a priori* estimates on such approximate solutions.

*Step 3.* Using the properties of compactness to extract a converging sub-sequence to pass to the limit.

*Step 4 and Step 5.* Using the monotonicity of the nonlinear operator  $\mathcal{A}(\mathbf{x})$  (See Appendix A) to prove that the limit process indeed leads to a weak solution.

The key idea of the proof is to find a solution to problem (4) when  $\phi$  is replaced by a Lipschitz function  $\phi_{reg}$  approximating  $\phi$ , such that  $|\phi_{reg}| \leq |\phi|$ . Throughout the paper we will refer to any of these approximations as *regular*  $\phi$  or  $\phi_{reg}$ , indistinctively. Then, we will show that a solution to problem (4) can be found as a limit of these *regularized* solutions. With these in mind, *Step 1*, *Step 3* and *Step 4* will be performed for any regular  $\phi_{reg}$ , the *a priori* estimates obtained in *Step 2* will be computed for  $\phi$  and thus, they will hold uniformly for any  $\phi_{reg}$ . The latter fact will allow us to find in *Step 5* a subsequence of *regularized* solutions that will converge to a solution of problem (4).

**Theorem 2.1.** *Let  $f$  and  $v_0$  satisfy*

$$v_0 \in L^{1+\eta}(\Omega) \quad \text{and} \quad f \in L^{(1+\gamma)^*}(0, T; L^{(1+\gamma)^*}(\Omega)), \quad (13)$$

*then, there exist a function  $v$  with the properties*

$$v \in L^{(1+\gamma)}(0, T; W_0^{1,(1+\gamma)}(\Omega)), \quad (14)$$

*and*

$$\phi(v)_t \in L^{(1+\gamma)^*}(0, T; W^{-1,(1+\gamma)^*}(\Omega)), \quad (15)$$

*such that it solves problem (4).*

*Proof.* For clarity we organize the proof in the steps previously described.

### Step 1. Approximate Solutions

Let  $\{w_j\}_{j=1}^\infty$  be a basis of  $V = W_0^{1,(1+\gamma)}(\Omega)$ . Construct the Faedo-Galerkin approximate solution of problem (4),  $v_m(t)$ , the following way. For any fixed  $t$

$$v_m(t) = \sum_{j=1}^m \zeta_j(t) w_j(x) \in [w_1, \dots, w_m] = \text{the space generated by } \{w_j\}_{j=0}^m$$

and satisfying

$$(\phi(v_m)_t, w_j) + \eta^\gamma \left( \frac{\nabla v_m}{|\nabla v_m|^{1-\gamma}}, \nabla w_j \right) = (f, w_j) \quad 1 \leq j \leq m, \quad (16)$$

$$v_m(0) = v_{0,m} \in [w_1, \dots, w_m],$$

where  $v_{0,m} \rightarrow v_0$  in  $L^{1+\eta}(\Omega)$ .

## Step 2. *A priori* Estimates

**Lemma 2.1.** *Set  $\phi(s) = s/|s|^{1-\eta}$ . Let  $v_m$  be a Faedo-Galerkin approximate solution of problem (4), then the following estimates hold.*

$$\sup_{0 \leq t \leq T} \|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} \leq C \left( \|v_0\|_{L^{1+\eta}(\Omega)}, \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}, T \right) \quad (17)$$

and

$$\|\nabla v_m\|_{L^{1+\gamma}(0,T;L^{1+\gamma}(\Omega))}^{1+\gamma} \leq C \left( \|v_0\|_{L^{1+\eta}(\Omega)}, \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}, T \right) \quad (18)$$

where  $(1+\eta)^* = (1+\eta)/\eta$ .

*Proof.* Multiply equation (16) by  $\zeta_j(t)$  and sum for  $1 \leq j \leq m$  to obtain (See Lemma A.2 in Appendix A for the first term)

$$\frac{d}{dt} \|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} + \frac{1+\eta}{\eta^{1-\gamma}} \int_{\Omega} |\nabla v_m|^{1+\gamma} = \frac{1+\eta}{\eta} (f, v_m) \quad (19)$$

and from Young's inequality

$$(f, v_m) \leq \frac{\eta}{1+\eta} \|f\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} + \frac{1}{1+\eta} \|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*}. \quad (20)$$

Now, since

$$\int_{\Omega} |\nabla v_m|^{1+\gamma} \geq 0$$

we get the inequality

$$\frac{d}{dt} \|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} \leq \|f\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} + \frac{1}{\eta} \|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*}.$$

Using Gronwall's lemma we get that for all  $t \in [0, T]$

$$\|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} \leq C \left( \|v_0\|_{L^{1+\eta}(\Omega)}, \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}, T \right)$$

which leads to the first estimate stated in (17).

*Note:* We have assumed, without loss of generality, that

$$\|v_{0,m}\|_{L^{1+\eta}(\Omega)} \leq \|v_0\|_{L^{1+\eta}(\Omega)}.$$

Integrating equation (19) in time

$$\|\phi(v_m)(T)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} + \frac{1+\eta}{\eta^{1-\gamma}} \int_0^T \int_{\Omega} |\nabla v_m|^{1+\gamma} = \frac{1+\eta}{\eta} \int_0^T (f, v_m) + \|v_{0,m}\|_{L^{1+\eta}(\Omega)}^{1+\eta}.$$

The above expression and inequality (20) imply that

$$\|\nabla v_m\|_{L^{1+\gamma}(0,T;L^{1+\gamma}(\Omega))}^{1+\gamma} \leq C \left( \|v_0\|_{L^{1+\eta}(\Omega)}, \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}, T \right)$$

which finishes the proof.  $\square$

**Remark 2.1.** Note that by the Poincaré inequality

$$\|v_m\|_{L^{1+\gamma}(0,T;L^{1+\gamma}(\Omega))} \leq C(\Omega) \|\nabla v_m\|_{L^{1+\gamma}(0,T;L^{1+\gamma}(\Omega))}$$

therefore the sequence  $\{v_m\} \subset L^{1+\gamma}(0, T; W_0^{1,1+\gamma}(\Omega))$  and it is uniformly bounded.

### Step 3. Passing to the limit

Let  $v_m(t)$  be the Faedo-Galerkin sequence of approximate solutions of problem (4) defined by (16). Estimates (17) and (18) in Lemma 2.1 imply that there exists a convergent subsequence  $\{v_\mu\}$  of  $\{v_m\}$  such that

$$v_\mu \rightharpoonup v \quad \text{in } L^{1+\gamma}(0, T; W_0^{1,1+\gamma}(\Omega)) \quad \text{weakly,} \quad (21)$$

$$\phi(v_\mu)(T) \rightharpoonup \xi \quad \text{in } L^{(1+\eta)^*}(\Omega) \quad \text{weakly,} \quad (22)$$

and as a consequence of (21) and the *Rellich-Kondrachov* compactness theorem we have that

$$v_\mu \rightarrow v \quad \text{in } L^{1+\gamma}(0, T; L^{1+\gamma}(\Omega)) \quad \text{strongly.} \quad (23)$$

In addition, inequality (18) implies

$$\frac{\nabla v_\mu}{|\nabla v_\mu|^{1-\gamma}} \rightharpoonup \chi \quad \text{in } L^{(1+\gamma)^*}(0, T; L^{(1+\gamma)^*}(\Omega)) \quad \text{weakly.} \quad (24)$$

Integrating equation (16) in time and using the aforementioned convergence results, we can take the limit as  $\mu \rightarrow \infty$  to find that for any  $w \in L^{1+\gamma}(0, T; W_0^{1,1+\gamma}(\Omega))$

$$\lim_{\mu \rightarrow \infty} \int_0^T (\phi(v_\mu)_t, w) = -\eta^\gamma \int_0^T (\chi, \nabla w) + \int_0^T (f, w). \quad (25)$$

We can conclude that

$$\phi(v_\mu)_t \rightharpoonup \vartheta \quad \text{in } L^{(1+\gamma)^*}(0, T; W^{-1, (1+\gamma)^*}(\Omega)) \quad \text{weakly,} \quad (26)$$

where the functional  $\vartheta$  is defined by the right hand side of equation (25). Note, that for any regular  $\phi$  the sequence  $\{\phi(v_\mu)_t\}$  lies in  $L^{(1+\gamma)^*}(0, T; W^{-1, (1+\gamma)^*}(\Omega))$  since  $\phi(v_\mu)_t$  inherits all the integrability properties of  $(v_\mu)_t$ . Using (23) and Theorem A.2 in Appendix A we can conclude that

$$\phi(v)_t = \vartheta. \quad (27)$$

Therefore, for any  $w \in L^{1+\gamma}(0, T; W_0^{1,1+\gamma}(\Omega))$

$$\int_0^T \langle \phi(v)_t, w \rangle = -\eta^\gamma \int_0^T (\chi, \nabla w) + \int_0^T (f, w). \quad (28)$$

Note also that  $L^{(1+\gamma)/\eta}(\Omega) \subset W^{-1, (1+\gamma)^*}(\Omega)$ , hence we have

$$\phi(v) \in L^{(1+\gamma)^*}(0, T; L^{(1+\gamma)/\eta}(\Omega)) \subset L^{(1+\gamma)^*}(0, T; W^{-1, (1+\gamma)^*}(\Omega)).$$

Using the previous fact, together with (26) and (27)

$$\phi(v) \in W^{1, (1+\gamma)^*}(0, T; W^{-1, (1+\gamma)^*}(\Omega)).$$

So by Theorem A.1 in Appendix A we conclude that

$$\phi(v) \in C([0, T]; W^{-1, (1+\gamma)^*}(\Omega))$$

and

$$\phi(v)(t) - \phi(v)(s) = \int_s^t \phi(v)_t \quad \text{for all } 0 \leq s \leq t \leq T. \quad (29)$$

Multiply equation (29) by  $w \in W^{1,1+\gamma}(\Omega)$  and integrate in  $\Omega$  to obtain

$$\begin{aligned} \langle \phi(v)(T) - \phi(v_0), w \rangle &= \int_0^T \langle \phi(v)_t, w \rangle \\ &= \lim_{\mu \rightarrow \infty} \int_0^T \langle \phi(v_\mu)_t, w \rangle \\ &= \lim_{\mu \rightarrow \infty} \langle \phi(v_\mu)(T) - \phi(v_{0,\mu}), w \rangle \\ &= \langle \xi - \phi(v_0), w \rangle. \end{aligned}$$

Since  $w$  is arbitrary, we conclude that

$$\phi(v)(T) = \xi. \quad (30)$$

#### Step 4. Monotonicity argument

It only remains to show that

$$\chi = \frac{\nabla v}{|\nabla v|^{1-\gamma}}$$

in equation (28). For that purpose, recall by the monotonicity Lemma A.1 in Appendix A that for any  $w \in L^{1+\gamma}(0, T; W_0^{1,1+\gamma}(\Omega))$

$$X_\mu \equiv \eta^\gamma \int_0^T \left( \frac{\nabla v_\mu}{|\nabla v_\mu|^{1-\gamma}} - \frac{\nabla w}{|\nabla w|^{1-\gamma}}, \nabla v_\mu - \nabla w \right) \geq 0$$

which we can rewrite as

$$X_\mu = T_{1,\mu} + T_{2,\mu}$$

where

$$T_{1,\mu} = \eta^\gamma \int_0^T \left( \frac{\nabla v_\mu}{|\nabla v_\mu|^{1-\gamma}}, \nabla v_\mu \right)$$

and

$$T_{2,\mu} = -\eta^\gamma \int_0^T \left( \frac{\nabla v_\mu}{|\nabla v_\mu|^{1-\gamma}}, \nabla w \right) - \eta^\gamma \int_0^T \left( \frac{\nabla w}{|\nabla w|^{1-\gamma}}, \nabla v_\mu - \nabla w \right).$$

Note that

$$\limsup_\mu X_\mu = \limsup_\mu T_{1,\mu} + \limsup_\mu T_{2,\mu} \geq 0. \quad (31)$$

From (21) and (24) one can easily see that

$$\limsup_\mu T_{2,\mu} = \lim_\mu T_{2,\mu} = -\eta^\gamma \int_0^T (\chi, \nabla w) - \eta^\gamma \int_0^T \left( \frac{\nabla w}{|\nabla w|^{1-\gamma}}, \nabla v - \nabla w \right). \quad (32)$$

For the term  $T_{1,\mu}$  one needs to be more careful. Using equation (16)

$$\begin{aligned} T_{1,\mu} &= - \int_0^T \langle \phi(v_\mu)_t, v_\mu \rangle + \int_0^T \langle f, v_\mu \rangle \\ &= - \frac{\eta}{\eta+1} \int_0^T \frac{d}{dt} \|\phi(v_\mu)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} + \int_0^T \langle f, v_\mu \rangle \\ &= \frac{\eta}{\eta+1} \|\phi(v_{0,\mu})\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} - \frac{\eta}{\eta+1} \|\phi(v_\mu)(T)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} + \int_0^T \langle f, v_\mu \rangle. \end{aligned}$$

Since by (30) and a well know property of weak limits

$$\|\phi(v)(T)\|_{L^{(1+\eta)^*}(\Omega)} = \|\xi\|_{L^{(1+\eta)^*}(\Omega)} \leq \liminf_{\mu} \|\phi(v_{\mu})(T)\|_{L^{(1+\eta)^*}(\Omega)}.$$

Thus, we are lead to

$$\limsup_{\mu} T_{1,\mu} \leq \frac{\eta}{\eta+1} \left( \|\phi(v_0)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} - \|\phi(v)(T)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} \right) + \int_0^T (f, v).$$

Now, substitute  $v$  for  $w$  in (28). Perform the integration in time to find that

$$\eta^{\gamma} \int_0^T (\chi, \nabla v) = \frac{\eta}{\eta+1} \left( \|\phi(v_0)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} - \|\phi(v)(T)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} \right) + \int_0^T (f, v). \quad (33)$$

Thus, from (31), (32) and (33) we observe that

$$\int_0^T \left( \chi - \frac{\nabla w}{|\nabla w|^{1-\gamma}}, \nabla v - \nabla w \right) \geq 0$$

if we choose  $w = v - \lambda\psi$  for  $\lambda > 0$  and  $\psi \in L^{1+\gamma}(0, T; W_0^{1,1+\gamma}(\Omega))$  in the previous equation, then

$$\int_0^T \left( \chi - \frac{\nabla(v - \lambda\psi)}{|\nabla(v - \lambda\psi)|^{1-\gamma}}, \nabla\psi \right) \geq 0.$$

Taking the limit as  $\lambda \rightarrow 0$  we finally obtain that

$$\int_0^T \left( \chi - \frac{\nabla v}{|\nabla v|^{1-\gamma}}, \nabla\psi \right) \geq 0$$

which implies by Lebesgue's lemma that

$$\chi = \frac{\nabla v}{|\nabla v|^{1-\gamma}}.$$

The previous fact completes the proof of Theorem 2.1 for any  $\phi_{reg}$ .

### Step 5. Going from $\phi_{reg}$ to $\phi$

Next, take  $\{\phi_k\}_{k=1}^{\infty}$  to be a sequence of regularized functions converging uniformly to  $\phi(s) = s/|s|^{1-\eta}$ . Then, *a priori* estimates (17) and (18), which are independent of  $k$ , hold for the sequences  $\{\phi_k(v_k)\}$  and  $\{v_k\}$ . Whence, *Step 3* and *Step 4* can be identically performed to find that  $v$  defined as

$$v = \lim_{k \rightarrow \infty} v_k$$

is a weak solution of the problem for the non regular  $\phi$ . □

**Corollary 2.1.** *There exists a weak solution to problem (5), where the gradient of  $u$  is understood as the pointwise gradient.*

*Proof.* Let  $v$  be a weak solution of problem (4) with initial condition  $v_0 = \phi^{-1}(u_0)$  and let  $u = \phi(v)$ . Immediately, the following holds:

(i)  $u = 0$  in  $(0, T) \times \partial\Omega$ ,

$$(ii) \quad u(0) = \phi(v(0)) = \phi(v_0) = \phi(\phi^{-1}(u_0)) = u_0,$$

$$(iii) \quad \phi(v)_t = u_t.$$

It only remains to show that the weak gradient of  $v$  and the pointwise gradient of  $u$  are related by

$$(iv) \quad \nabla v = (\phi^{-1})'(u)\nabla u \quad \text{a.e. in } (0, T) \times \Omega.$$

For this purpose, observe that since  $v \in L^{1+\gamma}(0, T; W_0^{1,(1+\gamma)}(\Omega))$  there exists a sequence  $v_m \in L^{1+\gamma}(0, T; C^\infty(\Omega))$  such that

$$v_m \rightarrow v \quad \text{strongly in } L^{1+\gamma}(0, T; L^{1+\gamma}(\Omega)) \quad \text{and a.e. in } (0, T) \times \Omega.$$

Define the sequence  $u_m = \phi(v_m)$ . Since  $v_m \in L^{1+\gamma}(0, T; C^\infty(\Omega))$ , we have that the following relation holds true a.e.

$$\nabla u_m = \begin{cases} \phi'(v_m)\nabla v_m & \text{if } |v_m| > 0 \\ 0 & \text{if } v_m = 0. \end{cases}$$

Therefore,

$$u_m \rightarrow u \quad \text{and} \quad \nabla u_m \rightarrow \nabla u \quad \text{a.e. in } (0, T) \times \Omega.$$

In addition,  $v_m = \phi^{-1}(u_m)$ , thus

$$\nabla v_m = (\phi^{-1})'(u_m) \nabla u_m \quad \text{a.e. in } (0, T) \times \Omega.$$

Sending  $m \rightarrow \infty$  in the previous expression, we find that

$$\nabla v = (\phi^{-1})'(u) \nabla u \quad \text{in } L^{1+\gamma}(0, T; L^{1+\gamma}(\Omega)).$$

To conclude the proof, substitute (iii) and (iv) in equation (9) to obtain equation (11).  $\square$

**Remark 2.2.** As pointed out in equation (6) and (8) an immediate consequence of Corollary 2.1 is that if  $u$  is a nonnegative solution of problem (4) then it solves problem (5) in the sense of Definition 1.2.

**Corollary 2.2.** *Let  $v$  a weak solution of the initial/boundary value problem (4). Then for any  $w \in L^{1+\gamma}(0, T; W_0^{1,(1+\gamma)}(\Omega))$*

$$\langle \phi(v)_t, w \rangle + \left( \frac{\nabla v}{|\nabla v|^{1-\gamma}}, \nabla w \right) = (f, w) \quad \text{a.e. in } [0, T].$$

*Proof.* Fix  $w \in L^{1+\gamma}(0, T; W_0^{1,(1+\gamma)}(\Omega))$  and let  $\{w_j\}$  be a basis for  $W_0^{1,(1+\gamma)}(\Omega)$ . Take a sequence  $\{\psi_m\}$  of the form

$$\psi_m = \sum_{j=1}^m d_j^m(t) w_j \quad \text{with } d_j^m(t) \in L^\infty([0, T])$$

such that  $\psi_m \rightarrow w$  strongly in  $L^{1+\gamma}(0, T; W_0^{1,(1+\gamma)}(\Omega))$ . This is possible by density of such finite sums in the mentioned space.

Since  $v$  is weak solution of problem (4) we get

$$\langle \phi(v)_t, \psi_m \rangle + \left( \frac{\nabla v}{|\nabla v|^{1-\gamma}}, \nabla \psi_m \right) = (f, \psi_m) \quad \text{a.e. in } [0, T].$$

Send  $m \rightarrow +\infty$  to conclude.  $\square$

### 3 Regularity

In this section we investigate basic regularity properties of solutions found in the existence Theorem 2.1. It is desirable to find more information on the time derivative of the function  $\phi(v)$ , in particular, it is worthwhile to find that it is a regular distribution.

**Theorem 3.1.** *Assume*

$$v_0 \in W_0^{1,1+\gamma}(\Omega), \quad \text{and} \quad f \in L^{(1+\eta)^*}(0, T; L^{(1+\eta)^*}(\Omega)).$$

Let  $v$  a solution of problem (4) constructed as in Theorem 2.1, then

$$(i) \quad v \in L^\infty(0, T; W_0^{1+\gamma}(\Omega)),$$

$$(ii) \quad v_t \text{ exists as a regular distribution that lies in } L^{1+\eta}(0, T; L^{1+\eta}(\Omega))$$

with the estimate

$$\int_{\{v>0\}} \left( \phi'(v)^{1/2} v_t \right)^2 + \sup_{[0, T]} \|\nabla v(t)\|_{L^{1+\gamma}(\Omega)}^{1+\gamma} \leq C \left( T, \|f\|_{L^{(1+\eta)^*}(0, T; L^{(1+\eta)^*}(\Omega))}, \|v_0\|_{W_0^{1,1+\gamma}(\Omega)} \right). \quad (34)$$

Moreover, when  $\phi$  is regular then  $\phi(v)_t$  also lies in  $L^{1+\eta}(0, T; L^{1+\eta}(\Omega))$  and

$$\phi(v)_t = \phi'(v)v_t. \quad (35)$$

*Proof.* Let  $\{\phi_k\}_{k=1}^\infty$  be a sequence of regularized functions converging uniformly to  $\phi(s) = s/|s|^{1-\eta}$  and let  $v_k(t)$  be the solution associated each  $\phi_k$ . Then

$$(\phi_k(v_k)_t, (v_k)_t) + \eta^\gamma \left( \frac{\nabla v_k}{|\nabla v_k|^{1-\gamma}}, \nabla(v_k)_t \right) = (f, (v_k)_t).$$

Hence,

$$\left\| \frac{\phi_k(v_k)_t}{\phi_k'(v_k)^{1/2}} \right\|_{L^2(\Omega)}^2 + \frac{\eta^\gamma}{1+\gamma} \frac{d}{dt} \|\nabla v_k\|_{L^{1+\gamma}(\Omega)}^{1+\gamma} = (f, (v_k)_t).$$

In addition, note that

$$(f, (v_k)_t) \leq 1/2 \left\| \frac{f}{\phi_k'(v_k)^{1/2}} \right\|_{L^2(\Omega)}^2 + 1/2 \left\| \frac{\phi_k(v_k)_t}{\phi_k'(v_k)^{1/2}} \right\|_{L^2(\Omega)}^2.$$

Thus, combining the last two relations we get

$$1/2 \left\| \frac{\phi_k(v_k)_t}{\phi_k'(v_k)^{1/2}} \right\|_{L^2(\Omega)}^2 + \frac{\eta^\gamma}{1+\gamma} \frac{d}{dt} \|\nabla v_k\|_{L^{1+\gamma}(\Omega)}^{1+\gamma} \leq 1/2 \left\| \frac{f}{\phi_k'(v_k)^{1/2}} \right\|_{L^2(\Omega)}^2. \quad (36)$$

Integrating (36) in time from 0 to  $T$ , we obtain that

$$1/2 \int_0^T \left\| \frac{\phi_k(v_k)_t}{\phi_k'(v_k)^{1/2}} \right\|_{L^2(\Omega)}^2 + \frac{\eta^\gamma}{1+\gamma} \sup_{[0, T]} \|\nabla v_k(t)\|_{L^{1+\gamma}(\Omega)}^{1+\gamma} \leq 1/2 \int_0^T \left\| \frac{f}{\phi_k'(v_k)^{1/2}} \right\|_{L^2(\Omega)}^2 + \|\nabla v_0\|_{L^{1+\gamma}(\Omega)}^{1+\gamma}. \quad (37)$$

By the hypothesis imposed on  $f$ , the right hand side of (37) converges to

$$1/2 \int_0^T \left\| \frac{f}{\phi'(v)^{1/2}} \right\|_{L^2(\Omega)}^2 + \|\nabla v_0\|_{L^{1+\gamma}(\Omega)}^{1+\gamma} \quad \text{as } k \rightarrow \infty.$$

This immediately implies that the right hand side is bounded. Because of the nonlinearities that occur in the left hand side of (37), it is not straightforward to send  $k \rightarrow \infty$  to establish estimate (34). For this purpose, we will first establish a weak convergence result for the sequence  $\{(v_k)_t\}$  in the following way.

Observe that since  $\phi(v_k)_t = \phi'_k(v_k)(v_k)_t$  then

$$\int_0^T \|(v_k)_t\|_{L^{1+\eta}(\Omega)}^{1+\eta} \leq \frac{1+\eta}{2} \int_0^T \left\| \frac{\phi_k(v_k)_t}{\phi'_k(v_k)^{1/2}} \right\|_{L^2(\Omega)}^2 + \frac{1-\eta}{2} \int_0^T \|1/\phi'_k(v_k)\|_{L^q(\Omega)}^q \quad (38)$$

where  $q = (1+\eta)/(1-\eta)$ . Note that

$$\phi'(s) = \frac{\eta}{|s|^{1-\eta}}, \quad (39)$$

therefore

$$\frac{1}{\phi'(s)} = \frac{|\phi(s)|^{\frac{1-\eta}{\eta}}}{\eta} \quad \text{and} \quad \int_0^T \|1/\phi'_k(v_k)\|_{L^q(\Omega)}^q = \frac{1}{\eta^q} \|\phi_k(v_k)\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}^{(1+\eta)^*}. \quad (40)$$

Hence, as a consequence of (37), (38), and (40) the sequence  $\{(v_k)_t\}$  is bounded in  $L^{1+\eta}(0, T; L^{1+\eta}(\Omega))$ . Thus, there exists a subsequence of  $\{(v_k)_t\}$ , labeled with the index  $\mu$  such that

$$(v_\mu)_t \rightharpoonup v_t \quad \text{weakly in } L^{1+\eta}(0, T; L^{1+\eta}(\Omega)) \quad \text{as } \mu \rightarrow +\infty. \quad (41)$$

Second, define for all  $\epsilon > 0$  and  $m \geq 1$  the set

$$\Omega_{m,\epsilon} := \bigcap_{j \geq m}^{+\infty} \{[0, T] \times \Omega : |v_j| \geq \epsilon\}.$$

Thus,

$$\begin{aligned} \int_0^T \left\| \frac{\phi_\mu(v_\mu)_t}{\phi'_\mu(v_\mu)^{1/2}} \right\|_{L^2(\Omega)}^2 &= \int_0^T \left\| \phi'_\mu(v_\mu)^{1/2} (v_\mu)_t \right\|_{L^2(\Omega)}^2 \\ &\geq \int_{\Omega_{m,\epsilon}} \left( \phi'_\mu(v_\mu)^{1/2} (v_\mu)_t \right)^2. \end{aligned} \quad (42)$$

Now, in  $\Omega_{m,\epsilon}$  we have the bound  $\phi'_\mu(v_\mu)^{1/2} \leq \eta \epsilon^{\eta-1}$  for  $\mu \geq m$  and clearly,

$$\phi'_\mu(v_\mu)^{1/2} \rightarrow \phi'(v)^{1/2} \quad \text{a.e. in } \Omega_{m,\epsilon}.$$

Using this fact with (41) we obtain

$$\phi'_\mu(v_\mu)^{1/2} (v_\mu)_t \rightharpoonup \phi'(v)^{1/2} v_t \quad \text{weakly in } L^{1+\eta}(\Omega_{m,\epsilon}).$$

Therefore, taking  $\liminf_{\mu \rightarrow +\infty}$  in (42) and using the weakly lower semicontinuity property of convex functionals on  $L^p$  it follows that

$$\int_{\Omega_{m,\epsilon}} \left( \phi'(v)^{1/2} v_t \right)^2 \leq \liminf_{\mu \rightarrow +\infty} \int_0^T \left\| \frac{\phi_\mu(v_\mu)_t}{\phi'_\mu(v_\mu)^{1/2}} \right\|_{L^2(\Omega)}^2. \quad (43)$$

As  $v_j \rightarrow v$  a.e. in  $[0, T] \times \Omega$ , it follows that

$$\lim_{m \rightarrow \infty, \epsilon \rightarrow 0} \Omega_{m,\epsilon} = \{|v| > 0\}.$$

Hence, taking these limits in (43) we obtain

$$\int_{\{|v|>0\}} \left( \phi'(v)^{1/2} v_t \right)^2 \leq \liminf_{\mu \rightarrow +\infty} \int_0^T \left\| \frac{\phi_\mu(v_\mu)_t}{\phi'_\mu(v_\mu)^{1/2}} \right\|_{L^2(\Omega)}^2. \quad (44)$$

This takes care of the first term in (37). The second term of the left hand side is simpler to deal with. Note that by (37) there exist a subsequence of  $\{v_k\}$ , labeled again with the index  $\mu$ , such that

$$v_\mu \rightharpoonup \xi \text{ in } L^\infty(0, T; W_0^{1,1+\gamma}(\Omega)) \text{ weak}^*.$$

Since the sequence already converged weakly in  $L^{1+\gamma}(0, T; W_0^{1,1+\gamma}(\Omega))$  to  $v$ , we conclude that  $\xi = v$ . Therefore, we can take  $\liminf_{\mu \rightarrow +\infty}$  in (37) to obtain

$$\begin{aligned} 1/2 \int_{\{|v|>0\}} \left( \phi'(v)^{1/2} v_t \right)^2 + \frac{\eta^\gamma}{1+\gamma} \sup_{[0,T]} \|\nabla v(t)\|_{L^{1+\gamma}(\Omega)}^{1+\gamma} \\ \leq 1/2 \int_0^T \left\| \frac{f}{\phi'(v)^{1/2}} \right\|_{L^2(\Omega)}^2 + \|\nabla v_0\|_{L^{1+\gamma}(\Omega)}^{1+\gamma}. \end{aligned} \quad (45)$$

To get estimate (34), observe that using the first expression in (40) one can prove, using Hölder's inequality, that

$$\int_0^T \left\| \frac{f}{\phi'(v)^{1/2}} \right\|_{L^2(\Omega)}^2 \leq \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}^2 \|\phi(v)\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}^{\frac{1-\eta}{\eta}}$$

which together with estimate (17) prove (i), (ii) and estimate (34). Finally when  $\phi$  is regular, it is Lipschitz, then the chain rule formula in (35) follows by a standard result for Sobolev functions.  $\square$

**Corollary 3.1.** *Assume the conditions of Theorem 3.1. Then for any regular  $\phi$ ,*

$$\phi(v)_t \in L^2(0, T; L^2(\Omega)),$$

and the following estimate holds

$$\|\phi(v)_t\|_{L^2(0,T;L^2(\Omega))}^2 \leq C \left( T, \phi'(0), \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}, \|v_0\|_{W_0^{1,1+\gamma}(\Omega)} \right).$$

*Proof.* The conditions on any regular  $\phi$  imply that for any  $s \in \mathbb{R}$

$$1 \leq \frac{\phi'(0)}{\phi'(s)}$$

thus, after applying the chain rule (35) in Theorem 3.1, it follows that

$$\begin{aligned} \int_{\{|v|>0\}} \phi(v)_t^2 &= \int_{\{|v|>0\}} (\phi'(v)v_t)^2 \\ &\leq \phi'(0) \int_{\{|v|>0\}} (\phi'(v)^{1/2}v_t)^2. \end{aligned}$$

In addition, observe that in the set  $\{v = 0\}$  we have that  $\phi(v) = 0$ . Hence, a direct calculation shows that  $\phi(v)_t = 0$  in the interior of this set. But  $\phi(v)_t$  is measurable, therefore

$$\int_{\{v=0\}} \phi(v)_t^2 = 0.$$

Consequently,

$$\|\phi(v)_t\|_{L^2(0,T;L^2(\Omega))}^2 \leq \phi'(0) \int_{\{|v|>0\}} (\phi'(v)^{1/2}v_t)^2.$$

Using estimate (34) in Theorem 3.1 we conclude the proof.  $\square$

**Remark 3.1.** Corollary 3.1 shows that the solutions of problem (4) constructed as in Theorem 2.1 are *solutions fortes* in the sense of [2].

**Theorem 3.2.** *Assume  $v$  is a solution of problem (4) constructed as in Theorem 2.1, and additionally assume that*

$$v_0 \in L^\infty(\Omega) \quad \text{and} \quad f \in L^\infty(0, T; L^\infty(\Omega)),$$

then

$$\|v\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C (\|v_0\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(0,T;L^\infty(\Omega))}, T). \quad (46)$$

*Proof.* In order to find an  $L^\infty$  bound on  $v$ , we would like to uniformly control its  $L^p$  norms. For this purpose, the key idea would be to multiply equation (4) by the test function  $v/|v|^{1-a}$  for any  $a \geq 1$  and use Gronwall's Lemma to establish the result. However, for a fixed time  $t$ , the test function  $v/|v|^{1-a}$  does not necessarily belong to  $W_0^{1,1+\gamma}(\Omega)$ , so that we need to regularize it. For this end, let us introduce the family  $\{\rho_\delta(s)\}_{\delta>0}$  approximating the function  $s/|s|^{1-a}$

$$\rho_\delta(s) = \frac{1}{(1 + \delta|s|)^a} \frac{s}{|s|^{1-a}}.$$

Note that  $\rho_\delta(v)(t) \in L^{1+\gamma}(0, T; W_0^{1,(1+\gamma)}(\Omega))$  since  $\rho_\delta(s)$  is a  $C^1([0, \infty))$  function with bounded derivative.

Using Corollary (2.2) we can chose  $\rho_\delta(v)$  as a test function in equation (4). Observe that for any regular  $\phi$ , the solution  $v$  has time derivative  $v_t \in L^{1+\eta}(0, T; L^{1+\eta}(\Omega))$  by Theorem (3.1), whence the chain rules applies,

$$\phi(v)_t = \phi'(v)v_t.$$

Therefore, the following relation holds immediately

$$\frac{d}{dt} \|\Phi_\delta(v)(t)\|_{L^1(\Omega)} = \langle \phi(v)_t, \Phi_\delta(v) \rangle$$

where

$$\Phi_\delta(s) = \int_0^s \phi'(z) \rho_\delta(z) dz.$$

Thus, we obtain

$$\frac{d}{dt} \|\Phi_\delta(v)(t)\|_{L^1(\Omega)} + \eta^\gamma (|\nabla v|^{1+\gamma}, \rho'_\delta(v)) = (f, \rho_\delta(v)). \quad (47)$$

The second term in the left hand side of (47) is nonnegative, thus the following inequality holds

$$\frac{d}{dt} \|\Phi_\delta(v)(t)\|_{L^1(\Omega)} \leq \|f\|_{L^\infty(\Omega)} \|\rho_\delta(v)\|_{L^1(\Omega)}.$$

Using the fact that

$$|\rho_\delta(s)| \leq 1 + \frac{\eta + a}{\eta} \Phi_\delta(s),$$

we obtain from the previous relation that

$$\frac{d}{dt} X_\delta(t) \leq \|f(t)\|_{L^\infty(\Omega)} \left( |\Omega| + \frac{\eta + a}{\eta} X_\delta(t) \right),$$

where

$$X_\delta(t) = \|\Phi_\delta(v)(t)\|_{L^1(\Omega)}.$$

Using Gronwall's lemma we get

$$X_\delta(t) \leq \exp\left(\frac{\eta + a}{\eta} \|f\|_{L^\infty(0,T;L^\infty(\Omega))} T\right) \{X_\delta(0) + \|f\|_{L^\infty(0,T;L^\infty(\Omega))} T\}. \quad (48)$$

Inequality (48) is valid for any  $\phi_{reg}$ , thus, it is also valid for  $\phi = s/|s|^{1-\eta}$ . Similarly, observe that

$$\Phi_\delta(v)(t) \longrightarrow \frac{\eta}{\eta + a} |v|^{\eta+a}(t) \text{ pointwise as } \delta \rightarrow 0 \text{ in } [0, T] \times \Omega.$$

Thus, sending  $\delta \rightarrow 0$  in (48) and using Fatou's Lemma it follows that

$$\frac{\eta}{\eta + a} \|v(t)\|_{L^{\eta+a}(\Omega)}^{\eta+a} \leq \exp\left(\frac{\eta + a}{\eta} \|f\|_{L^\infty(0,T;L^\infty(\Omega))} T\right) \left\{ \frac{\eta}{\eta + a} \|v_0\|_{L^{\eta+a}(\Omega)}^{\eta+a} + |\Omega| \|f\|_{L^\infty(0,T;L^\infty(\Omega))} T \right\}. \quad (49)$$

Taking the  $\eta + a$  root in (49) and letting  $a \rightarrow \infty$  we find that for  $0 \leq t \leq T$

$$\|v(t)\|_{L^\infty(\Omega)} \leq \exp(\eta^{-1} \|f\|_{L^\infty(0,T;L^\infty(\Omega))} T) \max(1, \|v_0\|_{L^\infty(\Omega)}) \quad (50)$$

which proves the result for any regular  $\phi$ . Next, take  $\{\phi_k\}_{k=1}^\infty$  to be a sequence of regularized functions converging uniformly to  $\phi(s) = s/|s|^{1-\eta}$ . Let  $v_k$  be the solution associated to each  $\phi_k$ , then, as in the proof of existence,

$$v = \lim_{k \rightarrow \infty} v_k \quad \text{pointwise in } (0, T) \times \Omega$$

Thus, estimate (50) holds for  $v$ . This concludes the proof.  $\square$

**Corollary 3.2.** *Assume  $u$  is a solution of problem (5) found as in Corollary 2.1, and additionally assume that*

$$u_0 \in L^\infty(\Omega) \quad \text{and} \quad f \in L^\infty(0, T; L^\infty(\Omega)),$$

then

$$\|u\|_{L^\infty(0, T; L^\infty(\Omega))} \leq C (\|u_0\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(0, T; L^\infty(\Omega))}, T).$$

*Proof.* The solution for problem (5) found as in Corollary 2.1 is given by  $u = \phi(v)$ , thus by estimate (46) we have

$$\|\phi^{-1}(u)\|_{L^\infty(0, T; L^\infty(\Omega))} \leq C (\|\phi^{-1}(u_0)\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(0, T; L^\infty(\Omega))}, T). \quad (51)$$

Since  $\phi^{-1}$  is a monotonically increasing function, we have the property that

$$\|\phi^{-1}(u)\|_{L^\infty(0, T; L^\infty(\Omega))} = \phi^{-1} (\|u\|_{L^\infty(0, T; L^\infty(\Omega))}).$$

Using the previous property, we can apply  $\phi$  on both sides of (51) to obtain

$$\begin{aligned} \|u\|_{L^\infty(0, T; L^\infty(\Omega))} &\leq \phi (C (\phi^{-1} (\|u_0\|_{L^\infty(\Omega)}), \|f\|_{L^\infty(0, T; L^\infty(\Omega))}, T)), \\ &\leq C (\|u_0\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(0, T; L^\infty(\Omega))}, T) \end{aligned}$$

which finishes the proof.  $\square$

## 4 Comparison Result, Uniqueness and Nonnegativity

Generally speaking, if  $v$  is a weak solution of problem (4) some basic regularity on  $\phi(v)_t$  must be obtained for pursuing a uniqueness result, otherwise this task can be very complex. Moreover, uniqueness may not be true. In Theorem (4.1) we will prove a comparison result due to Bamberger [2], that will lead to a uniqueness result under the assumption that

$$\phi(u)_t \in L^1(0, T; L^1(\Omega)). \quad (52)$$

In a hydrologic context, the previous assumption can be interpreted in the following way. Condition (52) implies that  $u_t \in L^1(0, T; L^1(\Omega))$  in problem (5). Hence

$$u \in C(0, T; L^1(\Omega)) \subseteq W^{1,1}(0, T; L^1(\Omega)).$$

Recall that when  $u$  is nonnegative,  $u$  represents the free water surface elevation, or the column of water at a given point in the domain  $\Omega$  in a physical system. Thus the volume  $\mathcal{V}$  of water in  $\Omega$  may be represented as

$$\mathcal{V}(\Omega, t) = \int_{\Omega} u(t).$$

Condition (52) implies that the the volume in the domain  $\Omega$  changes continuously in time. This is a natural condition when modeling hydrologic systems. The fact that the volume is a time continuous function follows when integrating expression (ii) of Theorem A.1 to obtain

$$\mathcal{V}(t_1) - \mathcal{V}(t_0) = \int_{t_0}^{t_1} \mathcal{V}_t(t),$$

where

$$\mathcal{V}_t(\Omega, t) = \int_{\Omega} u_t(t) \in L^1(0, T).$$

Therefore  $\mathcal{V}$  is an absolutely continuous function in  $[0, T]$ .

In the current section we will use the standard notation  $f^+$  and  $f^-$  to denote the positive and negative part of the function  $f$  respectively.

**Theorem 4.1** (Bamberger<sup>1</sup>). *Assume  $u$  and  $v$  are weak solutions of problem (4) associated to the initial data  $u_0$  and  $v_0$ , and the forcing terms  $f$  and  $g$  respectively. Assume the additional property that*

$$\langle \phi(u)_t, \phi(v)_t \rangle \in L^1(0, T; L^1(\Omega)) \quad \text{and} \quad f - g \in L^1(0, T; L^1(\Omega)), \quad (53)$$

then

$$\int_{\Omega} \lambda(\phi(u) - \phi(v)) \leq \int_{\Omega} \lambda(\phi(u_0) - \phi(v_0)) + \int_0^t \int_{\Omega} \lambda(f - g), \quad (54)$$

where  $\lambda(s)$  is any of the following three functions,  $|s|$ ,  $s^+$ , or  $s^-$ .

*Proof.* Since  $u$  and  $v$  are weak solutions of problem (4) then

$$\langle \phi(u)_t - \phi(v)_t, w \rangle + \left( \frac{\nabla u}{|\nabla u|^{1-\gamma}} - \frac{\nabla v}{|\nabla v|^{1-\gamma}}, \nabla w \right) = (f - g, w) \quad (55)$$

for any  $w \in W_0^{1, (1+\gamma)}(\Omega)$ . Let  $\{\beta_{\delta}(s)\}_{\delta>0}$  be the family of  $C^1(\mathbb{R})$  increasing functions such that,

(i)  $|\beta_{\delta}(s)| \leq 1$ , and

(ii)  $\beta_{\delta}(s) \rightarrow \lambda'(s)$  as  $\delta \rightarrow \infty$ .

Substituting  $w = \beta_{\delta}(u - v)$  in (55) we find that

$$\langle \phi(u)_t - \phi(v)_t, \beta_{\delta}(u - v) \rangle + \left( \frac{\nabla u}{|\nabla u|^{1-\gamma}} - \frac{\nabla v}{|\nabla v|^{1-\gamma}}, \beta'_{\delta}(u - v) \nabla(u - v) \right) = (f - g, \beta_{\delta}(u - v)).$$

Since  $\beta'_{\delta}(u - v) \geq 0$ , by Lemma A.1 in Appendix A, the second term in the previous expression is nonnegative, thus

$$\int_0^t \langle \phi(u)_t - \phi(v)_t, \beta_{\delta}(u - v) \rangle \leq \int_0^t (f - g, \beta_{\delta}(u - v)).$$

Note that  $\{\beta_{\delta}(u - v)\} \subset L^{\infty}(0, T; L^{\infty}(\Omega))$ . But  $\phi(u)_t$  and  $\phi(v)_t$  lie in  $L^{\infty}(0, T; L^{\infty}(\Omega))^*$  by assumption, thus

$$\langle \phi(u)_t - \phi(v)_t, \beta_{\delta}(u - v) \rangle = (\phi(u)_t - \phi(v)_t, \beta_{\delta}(u - v)).$$

Using Lebesgue's Dominated Convergence Theorem we can take the limit as  $\delta \rightarrow \infty$  in the above inequality to find that

$$\int_0^t (\phi(u)_t - \phi(v)_t, \lambda'(u - v)) \leq \int_0^t \int_{\Omega} \lambda(f - g).$$

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<sup>1</sup>See [2]

Observe that since  $\lambda'(u - v) = \lambda'(\phi(u) - \phi(v))$ , then for  $0 \leq t \leq T$ ,

$$\begin{aligned} \int_0^t (\phi(u)_t - \phi(v)_t, \lambda'(u - v)) &= \int_0^t ((\phi(u) - \phi(v))_t, \lambda'(\phi(u) - \phi(v))) \\ &= \int_0^t \frac{d}{dt} \int_{\Omega} \lambda(\phi(u) - \phi(v)) \\ &= \int_{\Omega} \lambda(\phi(u)(t) - \phi(v)(t)) - \int_{\Omega} \lambda(\phi(u_0) - \phi(v_0)), \end{aligned} \quad (56)$$

from which (60) follows.  $\square$

**Remark 4.1.** By hypothesis  $\phi(v) \in C([0, T]; L^1(\Omega))$  since  $\phi(v) \in W^{1,1}([0, T]; L^1(\Omega))$ . See Theorem A.1 in Appendix A. Thus, the last step in (56) can be safely performed.

**Remark 4.2.** Note that if  $\phi$  is regular we know from Corollary (3.1) that solutions of problem (4) constructed as in Theorem 2.1 satisfy

$$\phi(u)_t, \phi(v)_t \in L^2(0, T; L^2(\Omega)) \subset L^1(0, T; L^1(\Omega)).$$

Hence, the previous result applies for them.

**Corollary 4.1** (Uniqueness). *Assume  $u$  and  $v$  are weak solutions of problem (4) satisfying*

$$\phi(u)_t, \phi(v)_t \in L^1(0, T; L^1(\Omega)),$$

*then  $u = v$ .*

*Proof.* Use Theorem 4.1 with  $\lambda(s) = |s|$ ,  $u_0 = v_0$  and  $f = g$ .  $\square$

**Corollary 4.2.** *Assume  $u$  and  $v$  are weak solutions of problem (4) associated to the initial data  $u_0$  and  $v_0$ , and the forcing terms  $f$  and  $g$  respectively. Assume also*

$$\phi(u)_t, \phi(v)_t \in L^1(0, T; L^1(\Omega)) \quad \text{and} \quad f - g \in L^1(0, T; L^1(\Omega)), \quad (57)$$

*Additionally assume that*

$$\begin{aligned} v_0 &\leq u_0 \quad \text{a.e. in } \Omega, \\ g &\leq f \quad \text{a.e. in } (0, T) \times \Omega \end{aligned} \quad (58)$$

*then  $v \leq u$  a.e. in  $(0, T) \times \Omega$ .*

*Proof.* Use Theorem 4.1 with  $\lambda(s) = s^-$  to deduce that

$$\int_{\Omega} (\phi(u) - \phi(v))^- \leq 0,$$

thus,  $\phi(u) - \phi(v) \geq 0$  a.e. in  $(0, T) \times \Omega$ . Since  $\phi(s)$  is strictly increasing the result of the corollary follows.  $\square$

**Remark 4.3** (Nonnegativity for  $\phi_{reg}$ ). If we choose  $v_0 \equiv 0$  and  $g \equiv 0$  then  $v \equiv 0$  is the solution of problem (4). Using Corollary 4.2, for  $0 \leq u_0$  and  $0 \leq f$  we conclude that  $0 \leq u$  a.e. in  $(0, T) \times \Omega$ .

**Remark 4.4.** Note that when  $\phi$  is not regular and  $\phi(u)_t$  is not necessarily in  $L^1(0, T; L^1(\Omega))$ , however, the solution obtained in the proof of existence is nonnegative (for  $0 \leq u_0$  and  $0 \leq f$ ) since it is a pointwise limit of nonnegative solutions associated to regular  $\phi$ 's.

**Corollary 4.3.** *Assume  $u$  and  $v$  are weak solutions of problem (5) found as in Corollary 2.1, associated to the initial data  $u_0$  and  $v_0$ , and the forcing terms  $f$  and  $g$  respectively. Assume the additional property that*

$$u_t, v_t \in L^1(0, T; L^1(\Omega)) \quad \text{and} \quad f - g \in L^1(0, T; L^1(\Omega)), \quad (59)$$

then

$$\int_{\Omega} \lambda(u - v) \leq \int_{\Omega} \lambda(u_0 - v_0) + \int_0^t \int_{\Omega} \lambda(f - g), \quad (60)$$

where  $\lambda(s)$  is any of the following three functions,  $|s|$ ,  $s^+$ , or  $s^-$ .

The proof of Corollary 4.3 is an immediate consequence of Theorem 4.1 and it is an equivalent comparison result for problem (5). From this corollary, one obtains equivalent uniqueness and nonnegativity results for problem (5), thus concluding the proposed study of nonnegative solutions for problem (1) in a hydrological context.

#### 4.1 Final remarks.

Important issues to be addressed in future work should include:

- An appropriate study of existence and uniqueness of weak solutions of problem (1) when topographic effects are considered ( $z \neq 0$ ).
- Regularity of the free boundary for the two dimensional case both when  $z = 0$  (This would be an extension of the work of Esteban and Vázquez [6]), and  $z \neq 0$ .
- The connection between the regularity of the bathymetry  $z$  and the resulting weak solution of problem (1).
- Conditions for which the regularity in the time derivative can be improved as well as conditions for which the pointwise gradient can be bounded (for  $z = 0$ ).

## A Appendix

**Lemma A.1.** *The operator  $\mathcal{A}(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$  defined by*

$$\mathcal{A}(x) = \frac{x}{|x|^{1-\gamma}} \quad (61)$$

is monotone, i.e., for any  $x, y \in \mathbb{R}^n$

$$(\mathcal{A}(x) - \mathcal{A}(y)) \cdot (x - y) \geq 0.$$

*Proof.* Define the function  $\mathcal{B}(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$  as

$$\mathcal{B}(x) = |x|^{\gamma+1} \quad \text{where} \quad |x| = \left( \sum_{j=1}^n x_j^2 \right)^{\frac{1}{2}}$$

and note that

$$\frac{\partial}{\partial x_i} |x|^{\gamma+1} = (\gamma+1)|x|^{\gamma-1}x_i \quad \implies \quad \frac{1}{\gamma+1} \nabla \mathcal{B}(x) = \mathcal{A}(x).$$

Since  $\gamma+1 > 1$ , the function  $\mathcal{B}(x)$  is strictly convex. The gradient of a convex function is strictly increasing in each and all of its components, thus the result of the lemma holds true.  $\square$

**Lemma A.2.** *Set  $\phi(x) = x/|x|^{1-\eta}$ . Let  $v_m$  be a Faedo-Galerkin approximate solution of problem (4), then the following holds a.e. in  $(0, T) \times \Omega$*

$$\phi(v_m)_t = \begin{cases} \phi'(v_m)(v_m)_t & \text{if } |v_m| > 0 \\ 0 & \text{if } v_m = 0 \end{cases} \quad (62)$$

and

$$(\phi(v_m)_t, v_m) = \frac{\eta}{1+\eta} \frac{d}{dt} \|\phi(v_m)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} \quad (63)$$

*Proof.* Recall that

$$v_m(t) = \sum_{j=1}^m \zeta_j(t) w_j(x)$$

where the functions  $\{\zeta_j(t)\}$  are absolutely continuous functions since they solve a first order one dimensional ODE. The previous statement implies that the family of approximate solutions  $\{v_m(t)\}$  are absolutely continuous as well. Thus, to obtain formula (62) proceed as follows: in the set  $\{(t, x) : |v_m| > 0\}$  apply the chain rule, and in the set  $\{(t, x) : |v_m| > 0\}$  note that  $\phi(v_m)$  is equal to the constant zero, hence, in the interior of this set  $\phi(v_m)_t = 0$ .

In order to prove (63), observe on one hand that in the set  $\{(t, x) : |v_m| > 0\}$

$$\begin{aligned} \phi(v_m)_t v_m = \phi'(v_m)(v_m)_t v_m &= \frac{\eta}{1+\eta} (|v_m|^{1+\eta})_t \\ &= \frac{\eta}{1+\eta} (|v_m|^{\eta(1+\eta)^*})_t \\ &= \frac{\eta}{1+\eta} \frac{d}{dt} |\phi(v_m)|^{(1+\eta)^*}. \end{aligned}$$

On the other hand observe that in the set  $\{v = 0\}$  both terms in (63) are equal to 0.  $\square$

**Theorem A.1.** *(Calculus in abstract space) Let  $X$  a Banach space and let  $u \in W^{1,p}(0, T; X)$  for some  $1 \leq p \leq \infty$ . Then*

- (i)  $u \in C([0, T]; X)$  (after possibly being redefined on a set of measure zero), and
- (ii)  $u(t_1) = u(t_0) + \int_{t_0}^{t_1} u_t(\tau) d\tau$  for all  $0 \leq t_0 \leq t_1 \leq T$ .

*Proof.* See [7].  $\square$

Assume that  $\Omega$  is an open, bounded set, with smooth boundary, and  $T > 0$ . We have

**Theorem A.2.** *Let  $\psi$  a continuous real valued function, and  $0 < \eta \leq \gamma < 1$ . Assume that*

- (i)  $|\psi(x)| \leq |x|^\eta$

(ii)  $u_\mu \rightharpoonup u$  in  $L^{1+\gamma}(0, T; W^{1,1+\gamma}(\Omega))$

(iii)  $\psi(u_\mu)_t \rightharpoonup v$  in  $L^{(1+\gamma)^*}(0, T; W^{-1,(1+\gamma)^*}(\Omega))$

then,  $v = \psi(u)_t$ .

*Proof.* Set  $p = (1 + \gamma)/\eta$ . Note that by (i) we have

$$\|\psi(u_\mu)\|_{L^p(0,T;L^p(\Omega))}^p \leq \|u_\mu\|_{L^{1+\gamma}(0,T;L^{1+\gamma}(\Omega))}^{1+\gamma}. \quad (64)$$

Since  $L^p(0, T; L^p(\Omega))$  is a separable and reflexive Banach space (64) implies that

$$\psi(u_\mu) \rightharpoonup \xi \text{ weakly in } L^p(0, T; L^p(\Omega)). \quad (65)$$

Since

$$\frac{1+\gamma}{\eta} \geq \frac{1+\gamma}{\gamma} = (1+\gamma)^*,$$

it follows that

$$L^p(0, T; L^p(\Omega)) \subset L^{(1+\gamma)^*}(0, T; L^{(1+\gamma)^*}(\Omega)) \subset L^{(1+\gamma)^*}(0, T; W^{-1,(1+\gamma)^*}(\Omega)).$$

Then, for any  $\varphi \in C_c^1(0, T)$  and  $w \in W^{1,1+\gamma}(\Omega)$  we obtain

$$\int_0^T \langle \psi(u_\mu)_t, \varphi w \rangle = - \int_0^T (\psi(u_\mu), \varphi_t w). \quad (66)$$

Next, using (ii) and the Rellich-Kondrachow Compactness Theorem it is possible to obtain a subsequence  $\{u_{\mu'}\}$  of  $\{u_\mu\}$  such that

$$u_{\mu'} \rightarrow u \text{ strongly in } L^{1+\gamma}(0, T; L^{1+\gamma}(\Omega)).$$

Therefore,

$$\psi(u_{\mu'}) \rightarrow \psi(u) \text{ a.e. in } [0, T] \times \Omega.$$

Combine this with (65) to conclude that  $\xi = \psi(u)$ . In this way we can take  $\mu \rightarrow \infty$  in (66) to conclude that  $v = \psi(u)_t$ .  $\square$

**Theorem A.3.** *Assume that  $\Omega$  is measurable and  $|\Omega| < \infty$ . Assume also that  $f \in L^p(\Omega)$  for any  $1 \leq p < \infty$  and  $\|f\|_{L^p(\Omega)} \leq M$  for some  $M > 0$ . Then*

$$f \in L^\infty(\Omega) \quad \text{and} \quad \|f\|_{L^\infty(\Omega)} \leq M. \quad (67)$$

*Proof.* See [17, p. 126] for a version of this result. A slight modification of this proof will work for this version.  $\square$

## References

- [1] H. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. *Mathematische Zeitschrift*, 183:311–341, 1983.
- [2] A. Bamberger. Étude d'une équation doublement non linéaire. *Journal of Functional Analysis*, 24:148–155, 1977.

- [3] G.I. Barenblatt. On self-similar motions of compressible fluids in porous media. *Prikl. Math. Mech.*, 16:679–698, 1952. In Russian.
- [4] F. Bernis. Existence results for doubly nonlinear higher order parabolic equations on unbounded domains. *Mathematische Annalen*, 279:373–394, 1988.
- [5] D. Blanchard and G. Francfort. Study of a nonlinear heat equation with no growth assumptions on the parabolic term. *SIAM Journal of Mathematical Analysis*, 19(5):1032–1056, 1988.
- [6] J.R. Esteban and J.L. Vázquez. Homogeneous diffusion in  $\mathbb{R}$  with power-like nonlinear diffusivity. *Archive Rat. Mech. Analysis*, 103:39–80, 1988.
- [7] Lawrence C. Evans. *Partial Differential Equations*. American Mathematical Society, Providence, 2002.
- [8] P Di Giammarco, E. Todini, and P. Lamberti. A conservative finite elements approach to overland flow: the control volume finite element formulation. *Journal of Hydrology*, 175:267–291, 1996.
- [9] O. Grange and F. Mignot. Sur la résolution d’une équation et d’une inéquation paraboliques non linéaires. *Journal of Functional Analysis*, 11:77–92, 1972.
- [10] T.V. Hromadka, C.E. Berenbrock, J.R. Freckleton, and G.L. Guymon. A two-dimensional dam-break flood plain model. *Adv. Water Resources*, 8, 1985.
- [11] K. Ishige. On the existence of solutions of the Cauchy problem for a doubly nonlinear parabolic equation. *SIAM Journal on Mathematical Analysis*, 27(5):1235–1260, 1996.
- [12] J.L. Lions. *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*. Dunod Gauthier-Villars, Paris, 1969.
- [13] P.A. Raviart. Sur la résolution de certaines équations paraboliques non linéaires. *Journal of Functional Analysis*, 5:299–328, 1970.
- [14] M. Santillana and C. Dawson. Analysis of the continuous galerkin formulation to solve the diffusive wave approximation of the shallow water equations. In preparation.
- [15] J. L. Vázquez. *The Porous Medium Equation. Mathematical Theory*. Oxford University Press, USA, 2006.
- [16] C.B. Vreugdenhil. *Numerical Methods for Shallow-Water Flow*. Kluwer Academic Publishers, The Netherlands, 1998.
- [17] Richard L. Wheeden and Antoni Zygmund. *Measure and Integral: An Introduction to Real Analysis*. A series of Monographs and Textbooks. Marcel Dekker, Inc., New York, 1977.
- [18] Th. Xanthopoulos and Ch. Koutitas. Numerical simulation of a two dimensional flood wave propagation due to dam failure. *Journal of Hydraulic Research*, 14(4):321–331, 1976.
- [19] W. Zhang and T.W. Cundy. Modeling of two-dimensional overland flow. *Water Resources Res.*, 25:2019–2035, 1989.