M341 (56140), Sample Final Exam Questions

Below are some sample midterm questions. Please note that the intent of these is to help prepare for the exam, and that actual exam questions will not merely be modifications of these problems. Consult your HW, lecture notes, and books for additional sources of material to review. Finally, to obtain the full effect of an exam please complete these problems under time-pressure (150 minutes or less).

1. Let $\mathcal{V}$ be an n-dimensional vector space and $\mathcal{W}$ be an m-dimensional vector space.

   a) Suppose $n < m$. Show that there is no linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ such that L is onto.

   b) Suppose $n > m$. Show that there is no linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ such that L is one-to-one.

   c) Prove that $L\colon \mathcal{V} \to \mathcal{W}$ is an isomorphism only if $n = m$.

   d) Let $\mathcal{U}_3$ be the space of 3×3 upper triangular matrices and define the linear transformation $L\colon \mathcal{U}_3 \to \mathcal{M}_{33}$ by $L(A) = \frac{1}{2}\left(A + A^T\right)$. Is L onto? Is L one-to-one?

2. Let $\mathcal{V} = \mathcal{P}_2$ with standard basis $B = \left\{1,\, x,\, x^2\right\}$ and let $\mathcal{W} = \mathcal{M}_{22}$ with standard basis $D = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$. Consider $L\colon \mathcal{V} \to \mathcal{W}$ given by

$$L(p) = \begin{bmatrix} p(1) - p(0) & p(2) - p(0) \\ p(-1) - p(0) & p(-2) - p(0) \end{bmatrix}.$$

(For example, $L\left(x^2\right) = \begin{bmatrix} 1^2 - 0^2 & 2^2 - 0^2 \\ (-1)^2 - 0^2 & (-2)^2 - 0^2 \end{bmatrix}$.)

   a) Prove that L is a linear transformation.

   b) Find the matrix representation $[L]_{DB}$ of L with respect to the bases B and D.

   c) What is the dimension of $\mathrm{Ker}(L)$? Find a basis for $\mathrm{Ker}(L)$.

   d) What is the dimension of $\mathrm{Range}(L)$? Find a basis for $\mathrm{Range}(L)$.

   e) Verify the dimension theorem (i.e., rank-nullity theorem) for L.

3. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 7 \\ 5 & 10 & 13 & 18 \end{bmatrix}$ so that $\mathrm{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   a) Are the vectors $[1, 2, 5]^T$, $[2, 4, 10]^T$, $[3, 5, 13]^T$, $[4, 7, 18]^T$ linearly independent? Do they span $\mathbb{R}^3$?

   b) Find a basis for the span of the four vectors in part (a).

4. Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 8x_1 - 10x_2 \\ 3x_1 - 3x_2 \end{bmatrix}$. Define the standard basis $B = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ and an alternate basis $D = \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix}\right\}$. Consider a vector $\boldsymbol{v} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$.

a) Find the change of basis matrices P_{DB} (i.e., from B to D) and P_{BD} (i.e., from D to B).

b) Compute $[\boldsymbol{v}]_B$ and $[\boldsymbol{v}]_D$.

c) Find $[L]_{BB}$ and $[L]_{DD}$.

5. Consider $A = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$.

a) Write the characteristic polynomial $p_A(\lambda)$ and use this to determine the eigenvalues of A. [Hint: Factor out a term of the form $(\lambda - 1)$.]

b) Find the eigenspaces corresponding to the eigenvalues of A.

c) Is A diagonalizable? If so, find a nonsingular matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

6. Let $A = \begin{bmatrix} -8 & 4 & -3 & 2 \\ 2 & 1 & -1 & 0 \\ -3 & -5 & 4 & 0 \\ 2 & -4 & 3 & -1 \end{bmatrix}$.

a) Find the determinant of A using a cofactor expansion.

b) Find the determinant of A by using row operations to put A into upper triangular form. Verify that your answer agrees with part (a).

c) Is A nonsingular (i.e., invertible)?

7. The following two questions are unrelated to each other.

a) Show that $\mathcal{V} = \mathbb{R}$ with the usual operation of scalar multiplication but with addition given by $x \oplus y = 2(x + y)$ is not a vector space.

b) Consider the subset $\mathcal{S}$ of all matrices in $\mathcal{M}_{55}$ which have eigenvalue 1. Is $\mathcal{S}$ a *subspace* of $\mathcal{M}_{55}$? Explain why or why not.

8. True or false? Explain your answers.

a) The plane $x_1 + 3x_2 - 4x_3 = 1$ is a subspace of $\mathbb{R}^3$.

b) If A is a 3×5 matrix, then $\dim(\operatorname{Ker}(A)) \geq 2$.

c) Let $B = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ be a basis for a vector space $\mathcal{V}$. If n vectors $\{\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n\}$ span V then the coordinate vectors $\{[\boldsymbol{d}_1]_B, \ldots, [\boldsymbol{d}_n]_B\}$ are linearly independent.

d) Every linear transformation $L\colon \mathbb{R}^5 \to \mathbb{R}^4$ takes the form $L(\boldsymbol{x}) = A\boldsymbol{x}$ with A a 5×4 matrix.

e) Let $\mathrm{rref}(A)$ be the reduced row-echelon form of a matrix A. Then, the pivot columns of $\mathrm{rref}(A)$ form a basis of the column space of A (i.e., the span of the columns of A).

f) The vectors $\boldsymbol{b}_1 = 1 + t + 2t^2$, $\boldsymbol{b}_2 = 2 + 3t + 5t^2$, $\boldsymbol{b}_3 = 3 + 7 + 9t^2$ form a basis for $\mathcal{P}_2$.

g) [Harder...] The equation $p''(t) - p(t) = q(t)$ has a solution $p \in \mathcal{P}_3$ for any $q \in \mathcal{P}_3$.