M341 (56140), Sample Midterm #2 Solutions

1. Let $A = \begin{bmatrix} 4 & 3 & 1 & 2 \\ 1 & 9 & 0 & 2 \\ 8 & 3 & 2 & -2 \\ 4 & 3 & 1 & 1 \end{bmatrix}$.

a) Calculate the determinant of A using a cofactor expansion.

Solution: We expand $\det(A)$ about the third column:

$$\begin{aligned} \det(A) &= 1 \cdot \begin{vmatrix} 1 & 9 & 2 \\ 8 & 3 & -2 \\ 4 & 3 & 1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 4 & 3 & 2 \\ 1 & 9 & 2 \\ 4 & 3 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 4 & 3 & 2 \\ 1 & 9 & 2 \\ 8 & 3 & -2 \end{vmatrix} \\ &= -111 - 66 + 180 \\ &= 3. \end{aligned}$$

b) Recalculate the determinant using row reduction to verify your answer to (a).

Solution: To calculate the determinant, we can put A into upper triangular form using row operations as follows:

$$A = \begin{bmatrix} 4 & 3 & 1 & 2 \\ 1 & 9 & 0 & 2 \\ 8 & 3 & 2 & -2 \\ 4 & 3 & 1 & 1 \end{bmatrix} \xrightarrow[\langle 1\rangle \leftrightarrow \langle 2\rangle]{} \begin{bmatrix} 1 & 9 & 0 & 2 \\ 4 & 3 & 1 & 2 \\ 8 & 3 & 2 & -2 \\ 4 & 3 & 1 & 1 \end{bmatrix} \xrightarrow[\substack{\langle 2\rangle \leftarrow \langle 2\rangle - 4\langle 1\rangle \\ \langle 3\rangle \leftarrow \langle 3\rangle - 8\langle 1\rangle \\ \langle 4\rangle \leftarrow \langle 4\rangle - 4\langle 1\rangle}]{} \begin{bmatrix} 1 & 9 & 0 & 2 \\ 0 & -33 & 1 & -6 \\ 0 & -69 & 2 & -18 \\ 0 & -33 & 1 & -7 \end{bmatrix}$$

$$\xrightarrow[\substack{\langle 3\rangle \leftarrow \langle 3\rangle - 2\langle 2\rangle \\ \langle 4\rangle \leftarrow \langle 4\rangle - \langle 2\rangle}]{} \begin{bmatrix} 1 & 9 & 0 & 2 \\ 0 & -33 & 1 & -6 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & -1 \end{bmatrix} \xrightarrow[\langle 2\rangle \leftrightarrow \langle 3\rangle]{} \begin{bmatrix} 1 & 9 & 0 & 2 \\ 0 & -3 & 0 & -6 \\ 0 & -33 & 1 & -6 \\ 0 & 0 & 0 & -1 \end{bmatrix} \xrightarrow[\langle 3\rangle \leftarrow \langle 3\rangle - 11\langle 2\rangle]{} \begin{bmatrix} 1 & 9 & 0 & 2 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 1 & 60 \\ 0 & 0 & 0 & -1 \end{bmatrix} = U.$$

Therefore, $3 = \det(U) = (-1) \times (-1) \times \det(A)$ so $\det(A) = 3$ as expected.

c) What is the determinant of $-2A$? Why?

Solution: $\det(-2A) = (-2)^4 \det(A) = 16 \cdot 3 = 48$ since A has 4 rows.

2. Prove that if A is an orthogonal matrix (i.e., $A^T = A^{-1}$) then the determinant of A is either 1 or -1.

Solution: Since

$$\det(A) = \det\left(A^T\right) = \det\left(A^{-1}\right) = \frac{1}{\det(A)}$$

we have that $(\det(A))^2 = 1$, so $\det(A) = \pm 1$.

3. Let $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$.

a) Determine the eigenvalues of A.

Solution: The characteristic polynomial is

$$p_A(\lambda) = \det(A - \lambda I) = -\lambda^3 + \lambda = -\lambda(\lambda + 1)(\lambda - 1)$$

so the eigenvalues are $\lambda = 1, -1, 0$.

b) Find a nonsingular matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution: Computing the eigenspaces for each eigenvalue and putting the corresponding fundamental eigenvectors as the columns of a matrix P, we find that $A = PDP^{-1}$ with

$$P = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

c) Compute the determinant of A only using your answer to part (a) (i.e., do not compute the determinant directly.

[Hint: Recall the definition of the characteristic polynomial $p_A(\lambda)$.]

Solution: $\det(A) = p_A(0) = 0$.

4. The parts of the following question are unrelated.

a) Is $\mathcal{V} = \mathbb{R}$ with the usual scalar multiplication, but with addition defined as $\boldsymbol{x} \oplus \boldsymbol{y} = 3(x + y)$ a vector space? Justify your answer.

Solution: No. The operation $\oplus$ is not associative since

$$(\boldsymbol{x} \oplus \boldsymbol{y}) \oplus \boldsymbol{z} = 3(3(x+y)+z) = 9x + 9y + 3z \neq 3x + 9y + 9z = 3(x + 3(y+z)) = \boldsymbol{x} \oplus (\boldsymbol{y} \oplus \boldsymbol{z}).$$

b) Find the zero vector and the additive inverse of the vector space $\mathbb{R}^2$ with operations $[x, y] \oplus [w, z] = [x + w + 3, y + z - 4]$ and $a \odot [x, y] = [ax + 3a - 3, ay - 4a + 4]$.

Solution: $\mathbf{0} = 0 \odot [x, y] = [0x + 3(0) - 3, 0y - 4(0) + 4] = [-3, 4]$ while $-([x, y]) = [-x - 6, -y + 8]$.

c) If $\mathcal{V}$ is a vector space with subspace $\mathcal{W}_1$ and $\mathcal{W}_2$, prove that $\mathcal{W}_1 \cap \mathcal{W}_2$ is also a subspace.

[Hint: Do not forget to show that $\mathcal{W}_1 \cap \mathcal{W}_2$ is nonempty!]

Solution: Since the subspaces $\mathcal{W}_1$ and $\mathcal{W}_2$ both contain the zero vector, $\mathbf{0} \in \mathcal{W}_1 \cap \mathcal{W}_2$ and $\mathcal{W}_1 \cap \mathcal{W}_2$ is nonempty. Now suppose $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{W}_1 \cap \mathcal{W}_2$ and c is a scalar. Then $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{W}_1$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{W}_2$ so $\boldsymbol{x} + \boldsymbol{y} \in \mathcal{W}_1$ and $\boldsymbol{x} + \boldsymbol{y} \in \mathcal{W}_2$ since $\mathcal{W}_1$ and $\mathcal{W}_2$ are closed under vector addition. Therefore, $\boldsymbol{x} + \boldsymbol{y} \in \mathcal{W}_1 \cap \mathcal{W}_2$ and $\mathcal{W}_1 \cap \mathcal{W}_2$ is closed under vector addition as well. Similarly we find $\mathcal{W}_1 \cap \mathcal{W}_2$ is closed under scalar multiplication, so $\mathcal{W}_1 \cap \mathcal{W}_2$ is a subspace.

5. Consider $S = \{[2, -3, 4, -1]^T, [-6, 9, -12, 3]^T, [3, 1, -2, 2]^T, [2, 8, -12, 3]^T, [7, 6, -10, 4]^T\}$.

a) Is S linearly independent? If not, find a maximal linearly independent subset.

Solution: Let $A = \begin{bmatrix} 2 & -6 & 3 & 2 & 7 \\ -3 & 9 & 1 & 8 & 6 \\ 4 & -12 & -2 & -12 & -10 \\ -1 & 3 & 2 & 3 & 4 \end{bmatrix}$ be the matrix whose columns are vectors in S. Then $\text{rref}(A) = \begin{bmatrix} 1 & -3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, which does not have a pivot in each column so S is not linearly independent. One maximal linearly independent subset consists of the pivot columns of A—i.e., $B = \{[2, -3, 4, -1]^T, [3, 1, -2, 2]^T, [2, 8, -12, 3]^T\}$.

b) Does S span $\mathbb{R}^4$? If not, express span(S) in terms of a minimal spanning set.

Solution: No, S does not span $\mathbb{R}^4$ since rref(A) does not have a pivot in every row. A minimal spanning subset of S is the set B found in part (a), and $\text{span}(S) = \text{span}(B)$.

c) Construct a basis for span(S). What is $\dim(\text{span}(S))$?

Solution: B forms a basis for span(S), and $\dim(\text{span}(S)) = |B| = 3$.

d) Construct a basis for $\mathbb{R}^4$ that contains the maximal linearly independent subset found in part (a).

Solution: We must extend the linearly independent set B by adding to it another vector that is linearly independent to B. For example, let $\boldsymbol{v} = [1, 0, 0, 0]^T$ and define $\tilde{B} = B \cup \{\boldsymbol{v}\}$. Putting the vectors in $\tilde{B}$ as columns of a matrix $\tilde{A}$ we find that $\text{rref}\left(\tilde{A}\right) = I_4$ so $\tilde{B}$ is a basis of $\mathbb{R}^4$.

6. Prove that all vectors orthogonal to $[2, -3, 1]^T$ forms a subspace $\mathcal{W}$ of $\mathbb{R}^3$. What is $\dim(\mathcal{W})$ and why?

Solution: Let $\boldsymbol{v} = [2, -3, 1]^T$. Note that $\mathbf{0} \in \mathcal{W}$ since $\mathbf{0} \cdot \boldsymbol{v} = 0$ so $\mathcal{W}$ is nonempty. Now suppose $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{W}$ and c is a scalar. Then $(\boldsymbol{x} + \boldsymbol{y}) \cdot \boldsymbol{v} = (\boldsymbol{x} \cdot \boldsymbol{v}) + (\boldsymbol{y} \cdot \boldsymbol{v}) = 0 + 0 = 0$ and $(c\boldsymbol{x}) \cdot \boldsymbol{v} = c(\boldsymbol{x} \cdot \boldsymbol{v}) = c0 = 0$.

We will compute $\mathcal{W}$ explicitly in order to find its dimension. Since $\boldsymbol{x} = [x_1, x_2, x_3]^T \in \mathcal{W}$ if and only if $[2, -3, 1]^T \cdot \boldsymbol{x} = 2x_1 - 3x_2 + x_3 = 0$, we have that $x_3 = -2x_1 + 3x_2$ so $\boldsymbol{x} = x_1[1, 0, -2]^T + x_2[0, 1, 3]^T$. Therefore, $B = \left\{[1, 0, -2]^T, [0, 1, 3]^T\right\}$ is a basis for $\mathcal{W}$ and $\dim(\mathcal{W}) = 2$.