M341 (92150), Homework #7

Due: 10:00am, Friday, Aug. 02

Instructions: Questions are from the book "Elementary Linear Algebra, 4th ed." by Andrilli & Hecker. Please show all your work, not only your final answer, to receive credit. Keep answers organized in the same order the problems have been assigned.

Eigenvalues, eigenvectors, and diagonalization (3.4)

p. 191-196, #1(e), 3(b,e), 4(b,e), 5(b), 6, 7, 10(a), 12, 17, 21

In addition:

A) Recall the matrix $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$ considered in #3(b). We will determine how A transforms the unit circle in $\mathbb{R}^2$ using the following steps. This will give us a geometric interpretation of the eigenvalues and eigenspaces you found earlier.

a) The unit circle in $\mathbb{R}^2$ consists of all vectors $\boldsymbol{x}$ such that $\|\boldsymbol{x}\|^2 = 1$. Denoting $\boldsymbol{x} = [x_1, x_2]^T$ rewrite this as a formula in terms of x_1 and x_2.

b) Let $\boldsymbol{u} = A\boldsymbol{x}$ be the vector to which A maps $\boldsymbol{x}$. Denoting $\boldsymbol{u} = [u_1, u_2]^T$, write a formula satisfied by u_1 and u_2. [Hint: Since $A^{-1}\boldsymbol{u} = \boldsymbol{x}$, we know that $\|A^{-1}\boldsymbol{u}\|^2 = \|\boldsymbol{x}\|^2 = 1$. Therefore,

$$\begin{aligned} 1 = \|A^{-1}\boldsymbol{u}\|^2 &= (A^{-1}\boldsymbol{u}) \cdot (A^{-1}\boldsymbol{u}) \\ &= (A^{-1}\boldsymbol{u})^T (A^{-1}\boldsymbol{u}) \\ &= \boldsymbol{u}^T (A^{-1})^T A^{-1}\boldsymbol{u} \\ &= \boldsymbol{u}^T (AA^T)^{-1}\boldsymbol{u}. \end{aligned}$$

Now write $\boldsymbol{u}^T(AA^T)^{-1}\boldsymbol{u} = 1$ in terms of u_1 and u_2.]

c) Graph the two curves described by the formulas obtained above. Draw the vectors $\boldsymbol{x}_1 = \frac{1}{\sqrt{2}}[1, -1]^T$, $\boldsymbol{x}_2 = [1, 0]^T$, and $\boldsymbol{x}_3 = \frac{1}{\sqrt{2}}[1, 1]^T$ on top of the graph, along with the vectors $\boldsymbol{u}_1 = A\boldsymbol{x}_1$, $\boldsymbol{u}_2 = A\boldsymbol{x}_2$, and $\boldsymbol{u}_3 = A\boldsymbol{x}_3$. Among the three vectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$, which do not change direction under the transformation $\boldsymbol{x} \mapsto A\boldsymbol{x}$? [Hint: Use a graphing calculator or an online program like Wolfram Alpha to plot the curves.]

d) Decribe how the stretching of the unit circle seen in (c) relates to the eigenvalues and corresponding eigenspaces found earlier.