M341 (92150), Sample Midterm #1 Questions

Below are some sample midterm questions. Please note that the intent of these is to help prepare for the exam, and that actual exam questions will not merely be modifications of these problems. Consult your HW, lecture notes, and books for additional sources of material to review. Finally, to obtain the full effect of an exam please complete these problems under time-pressure (75 minutes or less).

1. Two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are said to be *parallel* if one is a scalar multiple of the other. Now consider the statement

"If $\|\boldsymbol{x}+\boldsymbol{y}\| \neq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$, then $\boldsymbol{x}$ is not parallel to $\boldsymbol{y}$."

a) Is this statement true or false? Justify your answer with a proof (if true) or a counterexample (if false).

b) State the contrapositive of the statement. Is the contrapositive true or false, and why?

c) State the converse and inverse of the statement. Are the converse and inverse true or false? Justify your answer with a proof or counterexample. [Hint: You may use the fact that if $\boldsymbol{x}\cdot\boldsymbol{y} = \|\boldsymbol{x}\|\|\boldsymbol{y}\|$ then $\boldsymbol{y}$ is a *positive* scalar multiple of $\boldsymbol{x}$, as we proved in lecture.]

2. Consider the matrix $A = \begin{bmatrix} 5 & -5 & 15 \\ 4 & -2 & -6 \end{bmatrix}$.

a) Find the complete solution set of $A\boldsymbol{x} = \boldsymbol{b}$ when $\boldsymbol{b} = [40, 19]^T$.

b) What is the reduced row echelon form of A?

c) What is the rank of A?

d) Without performing any extra computations, does the homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$ have nontrivial solutions? Why or why not?

3. Let A be an $n \times n$ lower triangular matrix, and B be an $n \times n$ upper triangular matrix. Suppose A and B have no zero entries on the main diagonal, and that AB is a diagonal matrix. Prove that A must be a diagonal matrix. [Hint: Use induction on j, where $1 \leq j \leq n$ represents the j^{th} column of A. Alternatively, try a proof by contradiction.]

4. Use the following steps to find the coefficients of the circle $x^2 + y^2 + ax + by = c$ that goes through the points $(5, -1), (6, -2), (1, -7)$.

a) Write a system of linear equations that must be satisfied by the coefficients.

b) Solve this system of equations to find the coefficients.

c) [Harder...] Given three distinct points in the plane, is it possible to have more than one circle go through all of them? Why or why not?

5. True or false? Justify your answers with a short proof or counterexample.

a) If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ then $A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$.

b) If A and B are $n \times n$ matrices such that $AB + BA = 0$, then $A^2 B^3 = B^3 A^2$.

c) The product of two skew-symmetric matrices is also skew-symmetric.

d) If $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors, then the projection of $\boldsymbol{x}$ onto $\boldsymbol{y}$ is orthogonal to the projection of $\boldsymbol{y}$ onto $\boldsymbol{x}$.

e) If the homogenous system $A\boldsymbol{x} = \boldsymbol{0}$ has nontrivial (i.e., nonzero) solutions, then the nonhomogenous system $A\boldsymbol{x} = \boldsymbol{b}$ may still have a unique solution for some choice of $\boldsymbol{b}$.