M346 (55820), Homework #11

Due: 12:00pm, Friday, Apr. 13

Adjoints

A) On $\mathbb{C}^3$ with the standard inner product, let $A\boldsymbol{x} = (5x_2 - ix_3, (2+3i)x_1 - 4x_2, ix_1 + x_3)^T$. Compute the adjoint A^* and determine $A^*\boldsymbol{x}$.

B) Let V be the space of smooth real-valued functions on $\mathbb{R}$ which vanish at infinity, equipped with inner product $\langle f | g \rangle = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2/2}dx$. Find the adjoints L_1^* and L_2^* of the linear operators $L_1 = d/dx$ and $L_2 = d^2/dx^2$.

C) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 2 & 4 & 3 \end{pmatrix}$. Find the matrix of the projection onto $\mathrm{Ker}(A)$. [Hint: Consider the projection onto $\mathrm{Ran}(A^*)$. Why?]

Self-adjoint and normal operators

A) Find an orthonormal basis consisting of eigenvectors of $A = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$.

B) Find an orthonormal basis consisting of eigenvectors of $A = \begin{pmatrix} 2 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 2 \end{pmatrix}$.

C) True or false? Justify your answers.

i. If L is self-adjoint, then L^k is self-adjoint.

ii. A normal operator with all eigenvalues real must be self-adjoint.

iii. The sum of two normal operators is normal.

iv. If $N \in M_{2,2}(\mathbb{R})$ is a real normal matrix then it must either be symmetric (and therefore self-adjoint) or take the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for some $a, b \in \mathbb{R}$.

D) *Every* matrix A has a (generally nonunique) Schur decomposition $A = UTU^{-1}$, where T is an upper triangular matrix and the columns of U are orthonormal.

i. Is $A = \begin{pmatrix} 5 & 2 \\ -7 & -4 \end{pmatrix}$ diagonalizable? Find the Jordan canonical form $A = PDP^{-1}$.

ii. Is A normal? Find a Schur decomposition $A = UTU^{-1}$.

iii. Why are your answers in (i) and (ii) different? Is it possible to find a Schur decomposition for A which coincides with the Jordan decomposition?