M346 (55820), Homework #12

Due: 12:00pm, Monday, Apr. 23

Isometries

A) Recall the Frobenius inner product $\langle A|B\rangle = \mathrm{Tr}(A^*B)$ for $A, B \in M_{m,n}(\mathbb{C})$. This defines the Frobenius norm $\|A\| = \sqrt{\langle A|A\rangle}$. Note that $\|A\|^2 = \mathrm{Tr}(A^*A) = \sum_{i=1}^n \sum_{j=1}^n |A_{ij}|^2$.

i. Show that the Frobenius norm is unitarily invariant. That is, show that if W is unitary then $\|WA\| = \|A\|$ and $\|AW\| = \|A\|$ for any A. [Hint: Use the cyclical properties of the trace: $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any A, B.]

ii. We say that A and B are unitarily equivalent if $A = UBU^*$ for some unitary U. In this case, the previous part implies that $\|A\| = \|B\|$. Use this to prove that the matrices $\begin{pmatrix} 1 & 2 \\ 2 & i \end{pmatrix}$ and $\begin{pmatrix} i & 4 \\ 1 & 1 \end{pmatrix}$ cannot be unitarily equivalent.

B) Is $A = \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix}$ unitary? Diagonalize A. [Hint: Remember that a matrix is unitary if and only if its columns are orthonormal!]

C) Let $\boldsymbol{v} \in \mathbb{C}^n$ with $\|\boldsymbol{v}\| = 1$, and define $H_{\boldsymbol{v}} = I - 2\boldsymbol{v}\boldsymbol{v}^*$ (this is known as a *Householder transformation* and reflects one vector to its negative while leaving its orthogonal complement invariant). Show that $H_{\boldsymbol{v}}$ is unitary.

Positive operators

A) Consider the symmetric matrix $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. Find $\sqrt{A}$ and verify directly that $\left(\sqrt{A}\right)^2 = A$.

Singular value decomposition (SVD)

A) Show that if A is positive, its spectral decomposition $A = UDU^*$ agrees exactly with its singular value decomposition $A = U\Sigma V^*$ (i.e., show that $\Sigma = D$ and $V = U$).

B) Let $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

i. Compute the SVD of A. Express your answer (i) as the sum of rank-1 terms and (ii) as $A = U\Sigma V^*$ for an appropriate U, V, and Σ.

ii. Find the best rank-2 approximation A_2 of A (where "best" implies closest to in squared Frobenius norm). Express your answer (i) as a sum of two rank-1 terms and (ii) as $A_2 = U\Sigma_2 V^*$ for an appropriate U, V, and Σ_2.

iii. Compute the approximation error $\|A - A_2\|$ in terms of the singular values of A.

C) Let $A = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}$.

i. Find the SVD of A.

ii. In $\mathbb{R}^2$, describe the image of the unit disc under the transformation A using SVD. That is, draw a picture of the region $\{A\boldsymbol{x} : \|\boldsymbol{x}\| \le 1\}$.

iii. Similarly, describe the inverse image of the unit disc by drawing a picture of the region $\{\boldsymbol{x} : \|A\boldsymbol{x}\| \le 1\}$.