M346 (55820), Homework #13

Due: 12:00pm, Friday, May 4

Infinite-dimensional inner product spaces (6.8)

A)

i. Let $\boldsymbol{v} = (a_1, a_2, a_3, \ldots)$ with $a_n = (-1)^n/\sqrt{n}$. Is $\boldsymbol{v}$ in $l_2(\mathbb{R})$?

ii. Consider the function $f(x) = 1/x^p$. For what $p \geq 0$ is f in $L_2([1, \infty))$? For what $p \geq 0$ is f in $L_2((0, 1])$?

Fourier series (6.9, 8.5, 8.7)

A) Use integration by parts to evaluate the following integrals with constant $k \neq 0$:

i. $\int_0^1 x \sin(kx) dx$

ii. $\int_0^1 x \cos(kx) dx$

iii. $\int_0^1 x \exp(ikx) dx$

[Hint: Use Euler's formula and your answers to (i) and (ii).]

B) Let $f(x) = x$ for $x \in [0, 1]$. Use your solutions from problem (A) for the following parts:

i. Write the Fourier sine series for f—that is, write

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$$

by finding the coefficients

$$c_n = 2\int_0^1 f(x) \sin(n\pi x) dx, \qquad n \in \{1, 2, 3, \ldots\}.$$

ii. Write the standard (real) Fourier series for f—that is, write

$$f(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} [\alpha_n \cos(2n\pi x) + \beta_n \sin(2n\pi x)]$$

by finding the coefficients

$$\alpha_n = 2\int_0^1 f(x) \cos(2n\pi x) dx, \quad \beta_n = 2\int_0^1 f(x) \sin(2n\pi x) dx, \qquad n \in \{0, 1, 2, \ldots\}.$$

iii. Write the standard (complex) Fourier series for f—that is, write

$$f(x) = \sum_{n=-\infty}^{\infty} \gamma_n \exp(2\pi i n x)$$

by finding the coefficients

$$\gamma_n = \int_0^1 f(x) \exp(-2\pi i n x) dx, \qquad n \in \mathbb{Z} = \{..., -1, 0, 1, ...\}.$$

iv. How fast do the coefficients decay in each of the three series expansions?

v. Check that the coefficients found in parts (ii) and (iii) satisfy

$$\alpha_n = \gamma_n + \gamma_{-n}, \quad \beta_n = i(\gamma_n - \gamma_{-n}), \qquad n \in \{0, 1, 2, ...\}.$$

[Note: This is due to the fact that the standard real Fourier series is simply a special case of complex Fourier series.]

vi. Choose any of the series computed above and plot the first few series approximations against the graph of $f(x)$. For which $x \in [0, 1]$ is your approximation most inaccurate, and why?

C) Show that if f is a real function, its Fourier coefficients in the standard complex Fourier series satisfy $\overline{\gamma_n} = \gamma_{-n}$.

D)

i. Derive a solution $u(x, t)$ to the partial differential equation (PDE)

$$\begin{aligned} \partial_t u &= -\partial_{xx} u \\ u(0, t) &= 0, \quad u(a, t) = 0 \qquad\qquad x \in [0, a], \quad t \geq 0 \\ u(x, 0) &= \begin{cases} x & \text{if } x < a/2 \\ a - x & \text{if } x \geq a/2 \end{cases} \end{aligned}$$

using Fourier sine series. How does the n^{th} Fourier coefficient evolve, and what does this imply about the solution $u(x, t)$ for arbitrarily small times $t > 0$? Contrast this behavior to that of the solution to the ordinary heat equation $\partial_t u = \partial_{xx} u$ discussed in class.

[Note: The equation above is called the *backward* heat equation because it arises from the ordinary heat equation under the time change $t \to -t$. It is *ill-posed* in that it behaves extremely badly for almost all initial conditions $u(x, 0)$.]

ii. Instead, solve the PDE

$$\begin{aligned} \partial_t u &= -\partial_{xx} u - \partial_{xxxx} u \\ u(0, t) &= 0, \quad u(a, t) = 0 \qquad\qquad x \in [0, a], \quad t \geq 0 \\ u(x, 0) &= \begin{cases} x & \text{if } x < a/2 \\ a - x & \text{if } x \geq a/2 \end{cases} \end{aligned}$$

using Fourier sine series. Now how does the n^{th} Fourier coefficient evolve and what does this imply about the solution $u(x, t)$ for small times $t > 0$? What happens as $t \to \infty$? Again, compare this to the ordinary heat equation.

[Note: By adding the term $-\partial_{xxxx} u$ to the equation, we have dramatically changed its behavior. This term, called a fourth-order regularization, overcomes the ill-posed nature of the term $-\partial_{xx} u$.]